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# SOLVING MULTIACTIVITY MULTIFACILITY CAPACITY-CONSTRAINED 0-1 ASSIGNMENT PROBLEMS

by

Krishan Lal Chhabra

Serial T-441
12 May 1981

The George Washington University
School of Engineering and Applied Science
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SOLVING A MULTIACTIVITY MULTIFACILITY

CAPACITY-CONSTRAINED 0-1 ASSIGNMENT PROBLEM

by

Krishan Lal Chhabra
B.M.E. 1965, University of Delhi
M.S. 1973, The George Washington University

A Dissertation submitted to

The Faculty of

The School of Engineering and Applied Science

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Dissertation directed by

Richard Martin Soland

Professor of Operations Research

Abstract

SOLVING A MULTIACTIVITY MULTIFACILITY
CAPACITY-CONSTRAINED 0-1 ASSIGNMENT PROBLEM

by

Krishan Lal Chhabra

Richard Martin Soland, Director of Research

A branch-and-bound solution algorithm and a computer program implementing this algorithm are developed to solve a multiactivity multifacility capacity constrained 0-1 assignment problem. The mathematical formulation for such a problem, called problem (P), is to find $x_{ij}$ and $y_k$ values that:

$$\text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_{ij} + \sum_{k=1}^{p} b_k y_k \tag{i}$$

$$\text{subject to} \quad \sum_{i=1}^{m} x_{ij} = 1 \qquad j=1,\ldots,n \tag{ii}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ijk} x_{ij} \leq s_k y_k \qquad k=1,\ldots,p \tag{iii}$$

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j \tag{iv}$$

$$y_k = 0 \text{ or } 1 \text{ for all } k \tag{v}$$

where $i$ , $j$ , $k$ are indices for designs, activities, and facilities, respectively; $x_{ij}$ has value 1 if and only if activity $j$ uses design $i$ , and $y_k$ has value 1 if and only if facility $k$ is used. A design involves the use of one or more facilities, and the same design may be used by several activities.

Problem (P) has the objective of minimizing the sum of $a_{ij}$'s -- the variable costs due to the assignments of activities to designs, and $b_k$'s -- the fixed costs due to the facilities used. Constraints (ii) and (iv) ensure that each activity is assigned to a single design. Each $d_{ijk}$ is the capacity required at facility $k$ if activity $j$ uses design $i$ , and is thus equal to zero if design $i$ does not involve the use of facility $k$ . Constraints (iii), therefore, ensure that for each facility $k$ used, the total capacity required does not exceed the capacity available $s_k$ . The difficulty in solving problem (P) stems from the indirect relationship between the assignments and facilities, i.e., an assignment $x_{ij} = 1$ bears on all the constraints (iii) for which $d_{ijk}$ is positive, and, therefore, on several $y_k$ variables.

The branch-and-bound solution algorithm uses Lagrangian relaxation as a basic step in obtaining lower bounds. In addition, it includes several operational rules, such as a branching rule for a judicious choice of the branching variable, a capacity rule to eliminate infeasible assignments, and a bounding rule to eliminate non-optimal assignments.

This dissertation includes relevant background leading to the formulation of problem (P), mathematical development of the branch-and-bound solution algorithm, a detailed test example, and computational test results using the computer program. The areas of application are identified, and suggestions for further improvement of the branch-and-bound solution algorithm are included.

The computer program has been written in FORTRAN IV. A detailed description of the computer program and guidelines for its use are included in a separate document entitled "Program Description and User's Guide for ZIPCAP--a Zero-one Integer Program to solve multiactivity multifacility Capacity-constrained Assignment Problems." Although developed for capacitated problems, the computer program can also be used to solve uncapacitated problems in which it is assumed that the facilities have infinite capacity.

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# 1. INTRODUCTION

Multiactivity multifacility assignment problems arise in such diverse areas as public health care systems and private multi-echelon inventory/distribution systems. Such systems involve the assignment of activities or tasks to groups of facilities in such a way that total system cost is minimized. The total system cost has components (fixed costs) that depend on the facilities actually used as well as components (variable costs) that depend solely on the assignment made. Most recently [Gross, Pinkus, and Soland (1979)] there has been interest in including facility capacity constraints as well. For this kind of problem, i.e., a multiactivity multifacility capacity-constrained 0-1 assignment problem, we have developed a solution algorithm of the branch-and-bound type and a computer program based upon it.

The computer program and guidelines for its use are described in a separate document [Chhabra and Soland (1980)] titled "Program Description and User's Guide for ZIPCAP -- a Zero-one Integer Program to solve multiactivity multifacility Capacity-constrained Assignment Problems."

This document describes the development of the solution algorithm and computational test results using the computer program. Suggestions for further improvement in the solution algorithm are also included.

This chapter reviews the relevant literature, provides background leading to the mathematical formulation of the multiactivity multifacility capacity-constrained 0-1 assignment problem, called problem (P), and includes potential areas of application. The theoretical base for developing the algorithm/methodology are described in Chapter 2. Various components of the methodology are covered in detail in Chapter 3. Chapter 4 provides an overview of the computational procedure and the computer program, whereas computational test results are given in Chapter 5. Suggestions for further research and potential improvements in the algorithm are included in Chapter 6.

It may be noted that the basic terminology, described below, in the formulation of problem (P) includes: activities that must be assigned, facilities which serve the activities, designs involving one or more facilities, fixed costs associated with the facilities, and variable costs associated with the assignment of activities to designs.

The following review of the relevant literature starts with the classical assignment problem and leads to the formulation of problem (P). Different authors have used various terminologies in describing relevant formulations. In the following discussion, the original terminologies are used, and are followed by our equivalent terminology, where appropriate, shown in parenthesis.

## 1.1 Generalized Assignment Problem

In a classical assignment problem [Hillier and Lieberman (1980)], the purpose is to find optimal pairs of agents and tasks or activities. Each task is assigned to a single agent, and each agent is given a single task, and the suitability of a particular set of assignments is determined by a single criterion function such as minimization of cost. In a generalized assignment problem (GAP), several tasks can be assigned each agent, subject to the resources available to the various agents [Ross and Soland (1975)], e.g., assigning software development tasks to programmers and assigning jobs to computers in a computer network.

A variety of well-known facility location and location-allocation problems have been shown to be equivalent to, and therefore solvable as GAP's [Ross and Soland (1977)]. Here, in general, the tasks represent demand centers for a good or service, and the agents represent supply centers to be established at potential sites or locations. Each demand center must be supplied from a supply center. A fixed cost is incurred for each supply center established, and, in addition, there is a cost incurred for each unit processed at a supply center and transportation costs incurred for the units sent from supply centers to demand centers. The problem may be "uncapacitated" -- when there is no limit to the number of units that may be processed by

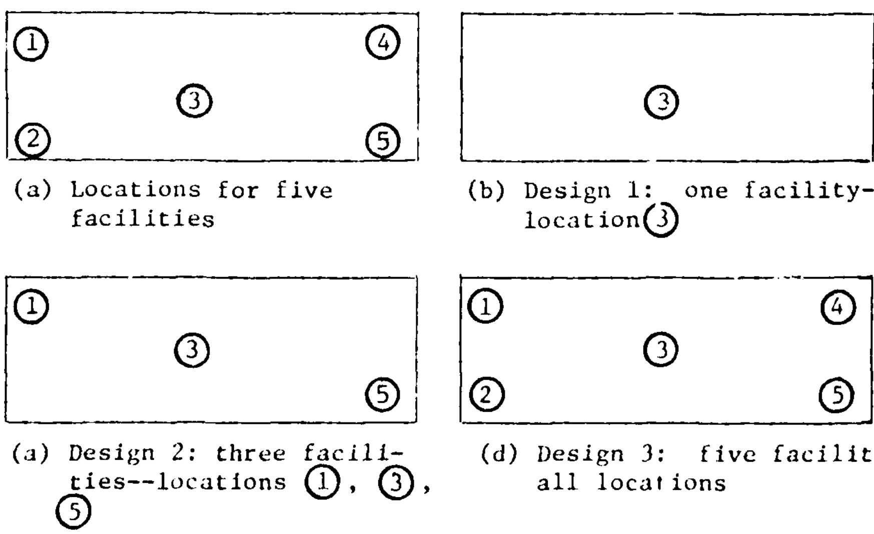
a supply center, or "capacitated" -- when there are restrictions on the number of units that may be processed. The objective is to select supply center locations and set up a distribution assignment so that the total cost is minimized.

## 1.2 Multiactivity Multifacility Uncapacitated Assignment Problem

A salient feature of the above facility location problems is that each demand center (activity) is assigned to a single supply center (facility). Sometimes, however, it may be desirable to assign an activity to more than one facility. This leads to the concept of "design," and the multiactivity multifacility assignment problem [Pinkus, Gross, and Soland (1973)]. Before describing such a problem, some terminology is considered first.

A design involves the use of one or more facilities, and represents a meaningful configuration of facilities along with a meaningful strategy for using them -- as illustrated in the following examples.

Consider five facilities and their locations as shown in Figure 1(a). (From practical considerations, these may be existing and/or potential locations.) Three of the possible designs are shown in Figures 1(b) to 1(d). Design 1 is completely centralized since it uses only one facility, whereas design 3 is completely decentralized since it uses all the facilities.




(a) Locations for five facilities

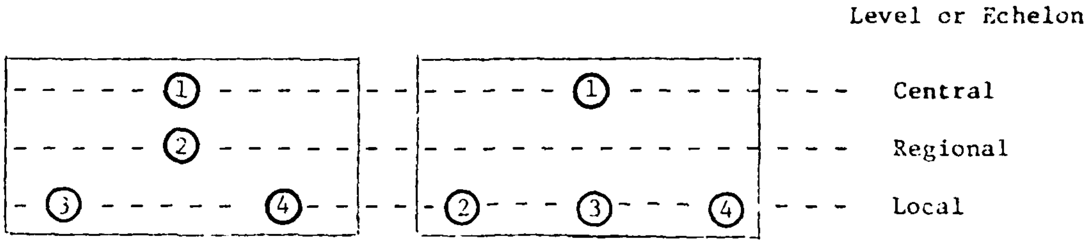
(b) Design 1: one facility--location 3

(a) Design 2: three facilities--locations 1, 3, 5

(d) Design 3: five facilities--all locations

Figure 1. Examples of alternate designs for a system of five facilities

It is possible for several designs to have the same facilities but different configuration and strategies for using these facilities, e.g., a multiproduct multi-echelon inventory system [Gross, Pinkus, and Soland (1979)]. Figure 2(a) shows design 1 containing certain facilities (warehouses) at the central, regional, and local levels or echelons. Figure 2(b) shows design 2 with the same facilities but having a different configuration.




(a) Design 1 (b) Design 2

Figure 2. Example of alternative designs having the same facilities but different configuration

The distribution of a given activity at various facilities under design 1 would be different than under design 2, depending, of course, on the inventory policies. This results in different variable costs (described later) for that activity under design 1 as against design 2. In fact, it is possible to have a situation where two or more designs have the same facilities and the same configuration but different strategies, resulting in different variable costs. For example, one strategy might specify an equal distribution of a specific activity over the various facilities, whereas another strategy could impose a different distribution scheme over the same facilities.

In general, if a system is to be composed of at most $p$ facilities, the number of alternative designs is $2^p-1$ if no two designs have the same facilities. However, with the same facilities but different configurations and strategies, the number of alternative designs could be much higher. In practice, it is possible to eliminate a majority of alternative designs because of geographical, political, economical, and other factors.

The multiactivity multifacility assignment problem seeks minimization of some measure of total system cost such as, total expected cost over a given time period or total discounted cost over the lifetime of the system. The system cost will include investment costs for building or leasing the system, operating costs for operation and maintenance of the system, and the costs for providing necessary services. Both the investment costs and the operating costs have fixed as well as variable components [Ross and Soland (1980)]. The fixed components include those costs associated with the facilities of a given design which are independent of the activities served. Such costs are called fixed costs. On the other hand, the variable components and the service costs include those costs which are completely dependent on the service demand of the activities at the various facilities in a given design. Such costs are called variable costs. By definition, both the fixed costs and the variable costs are relative terms.

An equivalent formulation of the multiactivity multifacility assignment problem defined by Pinkus, Gross, and Soland (1973) is as follows.

Let $a_{ij}$ = variable cost of activity $j$ using design $i$ $(i=1,\ldots,m;\ j=1,\ldots,n)$

$b_k$ = fixed cost of facility $k$ $(k=1,\ldots,p)$

$b_{ik}$ = 1 if facility $k$ is included in design $i$ ,

= 0 otherwise.

The decision variable $x_{ij}$ is defined as:

$$x_{ij} = 1 \text{ if activity } j \text{ uses design } i ,$$
$$= 0 \text{ otherwise.}$$

Then, the uncapacitated assignment problem called problem (PU) is to find $x_{ij}$ values that:

$$(\text{PU}) \begin{cases} \text{Minimize} & \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_{ij} + \sum_{k=1}^{p} b_k u\left(\sum_{i=1}^{m} b_{ik} \sum_{j=1}^{n} x_{ij}\right) \quad (1) \\ \text{subject to} & \sum_{i=1}^{m} x_{ij} = 1 \quad \text{for } j=1,\ldots,n \quad (2) \\ & x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j \quad (3) \\ \text{where} & u(\cdot) = 0 \text{ if } (\cdot) \leq 0 , \\ & \qquad = 1 \text{ if } (\cdot) > 0 . \end{cases}$$

The objective function of this problem consists of two distinct parts. The first part represents the total variable cost, and the second, the total fixed cost of the system. Constraints (2) and (3) ensure that each activity is assigned to a single design. Of course, the optimal solution may involve the use of more than one design.

Problem (PU) is a 0-1 nonlinear programming problem (because of the step function u ), and a branch-and-bound algorithm using linear underestimates for the nonlinear part of the objective function has been described in Pinkus, Gross, and Soland (1973). A heuristic procedure for this problem is given by Khumawala and Stinson (1980) in an unpublished paper. This procedure is an extension of some earlier work [Khumawala (1973)].

## 1.3 Adding Capacity Constraints -- Problem (P)

A weakness of problem (PU) is that it assumes unlimited capacity available at each facility in terms of the activities using a given facility. In practice, a facility may not have the capability to serve every activity, and may have restrictions as to the total capacity available to handle more than one activity.

Let $s_k$ = capacity available at facility $k$, and

$d_{ijk}$ = capacity required at facility $k$ for activity $j$ when activity $j$ uses design $i$.

If design $i$ does not include facility $k$, then $d_{ijk} = 0$ for all $j$.

Define the decision variable $y_k$ as:

$$y_k = 1 \text{ if facility } k \text{ is used,}$$
$$= 0 \text{ otherwise .}$$

Then the assignment problem [Gross, Pinkus, and Soland (1979)], called problem (P) is to find $x_{ij}$ and $y_k$ values that:

$$
(P)\left\{
\begin{array}{llll}
\text{Minimize} & \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x_{ij} + \sum_{k=1}^{p} b_k y_k & & (4) \\
\text{subject to} & \sum_{i=1}^{m} x_{ij} = 1 & j=1,\ldots,n & (2) \\
 & \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ijk}\, x_{ij} \le s_k y_k & k=1,\ldots,p & (5) \\
 & x_{ij},\ y_k = 0 \text{ or } 1 \text{ for all } i,j,k & & (6)
\end{array}
\right.
$$

Constraints (5) of problem (P) ensure that the capacities available at the facilities are not violated. Problem (P) is, thus, a multiactivity multifacility capacity-constrained 0-1 assignment problem, as compared to problem (PU) which is uncapacitated. In problem (P), constraints (2) along with the part of constraints (6) involving the $x_{ij}$'s ensure that each activity is assigned to a single design. Of course, the optimal solution may result in the use of more than one design.

For an example of five facilities and three designs as shown in Figures 1(b) to 1(d), and four activities; the matrix $[a_{ij}|b_k|d_{ijk}]$ is as shown in Figure 3.

### 1.3.1 Comparison with the uncapacitated assignment problem.

Comparison of the capacitated problem (P) with the uncapacitated problem (PU) shows that the objective functions (1) and (4) are equivalent and constraints (2) in each are the same. Constraints (5) serve to impose the capacity constraints and at the same time, for a given design, the relevant facilities are forced in the solution. For an

| | | Variable Costs ($a_{ij}$) | | | | Fixed Costs ($b_k$) | | | | | Capacities Required ($d_{ijk}$) e.g., for k=1* | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Activities (j) | | | | Facilities (k) | | | | | Activities (j) | | | |
| | | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 4 |
| Designs (i) | 1 | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | | | $b_3$ | | | 0 | 0 | 0 | 0 |
| | 2 | $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | $b_1$ | | $b_3$ | | $b_5$ | $d_{211}$ | $d_{221}$ | $d_{231}$ | $d_{241}$ |
| | 3 | $a_{31}$ | $a_{32}$ | $a_{33}$ | $a_{34}$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $d_{311}$ | $d_{321}$ | $d_{331}$ | $d_{341}$ |

Figure 3.

Matrix of variable costs, fixed costs, and capacities required -- example

*Similar $d_{ijk}$ values exist for $k=2,\ldots,5$ depending on the inclusion of the facility in a design.

$x_{ij}$ equal to 1, all the facilities with $d_{ijk} > 0$ must have $y_k$ values equal to 1 in order to satisfy (5) and the corresponding fixed costs $b_k$ are therefore included in (4). If $y_k = 0$ and $d_{ijk} > 0$, then $x_{ij}$ must be 0 in order to satisfy (5).

Problem (P) has been formulated as a 0-1 linear programming problem whereas problem (PU) was formulated as a 0-1 nonlinear programming problem.

Note that problem (PU) can be easily obtained as a special case of problem (P) by letting $d_{ijk}$ equal 1 (for all j) if design i uses facility k, and setting all $s_k$ equal to n. In other words, the corresponding formulation is to find $x_{ij}$ and $y_k$ values that:

$$
(\text{PI})\left\{
\begin{array}{lll}
\text{Minimize} & \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij} + \sum_{k=1}^{p} b_k y_k & (4) \\
\text{subject to} & \sum_{i=1}^{m} x_{ij} = 1 \qquad j=1,\dots,n & (2) \\
 & \sum_{i=1}^{m} e_{ik} \sum_{j=1}^{n} x_{ij} \le n y_k \qquad k=1,\dots,p & (7) \\
 & x_{ij}, y_k = 0 \text{ or } 1 \text{ for all } i,j,k & (6) \\
\text{where} & e_{ik} = 1 \text{ if design } i \text{ uses facility } k, & \\
 & \phantom{e_{ik}} = 0 \text{ otherwise}. &
\end{array}
\right.
$$

### 1.3.2 Comparison with the fixed-charge location-allocation problem.

Problem (P) bears a resemblance to the well-known fixed-charge location-allocation problem or capacitated facility location problem [Geoffrion (1975); Ross and Soland (1977)]. There are, however, very significant differences between the two. In order to point out these differences, here is a statement of the location-allocation problem (LA) as given by Gross, Pinkus, and Soland (1979) in a form

similar to that of problem (P).

Find $x_{kj}$ and $y_k$ values that

$$
\text{(LA)}\begin{cases}
\text{Minimize} & \sum_{k=1}^{p}\sum_{j=1}^{n} a_{kj}\, x_{kj} + \sum_{k=1}^{p} b_k y_k & (8)\\
\text{subject to} & \sum_{k=1}^{m} x_{kj} = 1 \qquad j=1,\dots,n, & (9)\\
& \sum_{j=1}^{n} d_j\, x_{kj} \le s_k y_k \qquad k=1,\dots,p & (10)\\
& x_{kj} \ge 0\ ,\ y_k = 0 \text{ or } 1 \text{ for all } j \text{ and } k & (11)
\end{cases}
$$

Here $x_{kj}$ represents the fraction of customer (activity) j's demand that is supplied by a facility at location k.

The most important distinction between problem (LA) and problem (P) is the relationship between assignments and facilities. In problem (LA) there is a direct connection between the assignments made and the facilities required, and each assignment affects only one facility, i.e., the assignment $x_{kj} > 0$ has a bearing on only one of the constraints (10) and, therefore, on only one variable $y_k$ . On the other hand, in problem (P), the connection between the assignments made and the facilities required is indirect, and each assignment can affect several facilities, i e., the assignment $x_{ij} = 1$ bears on all of the constraints (5) for which $d_{ijk} > 0$ and, therefore, on several variables $y_k$ .

Another distinction is the relative difficulty of the two problems. While problem (LA) is not easy to solve, branch-and-bound approaches have been successful in dealing with it because once values are specified for the $y_k$ , the $x_{jk}$ are found by solving a transportation problem. Problem (LA) becomes more difficult if the constraints $x_{kj} \ge 0$ in (10) are replaced by $x_{kj} = 0$ or 1 in order to preclude supply of customer (activity) j's demand by more than one facility. With this change,

problem (LA) may be treated as a generalized assignment problem and is solvable using an efficient branch-and-bound algorithm [Ross and Soland (1977)]. Problem (P) is more difficult than this variation of problem (LA) because of the above stated indirect connection between the assignments and the facilities. Even after values have been specified for all the $y_k$, problem (P) remains a difficult 0-1 linear programming problem because of the interaction of the constraints.

### 1.3.3 Solving problem (P).

The capacitated problem (P) has $mn+p$ 0-1 variables and $n+p$ constraints, so the problem dimensions may be large from practical considerations. For example, with $m=n=30$ and $p=20$, problem (P) has 920 variables and 50 constraints. The 0-1 LP computer codes generally available are limited in terms of problem size. For example, the code used by Gross, Pinkus, and Soland (1979) can handle up to 40 variables and 20 constraints. A better and more efficient code [Geoffrion and Nelson (1968)] allows up to 90 variables and 50 constraints. This fact, together with the structure of problem (P) suggests that a specialized algorithm could be developed that would be more efficient for practical problems than the general integer linear programming algorithms (on which the available codes are based).

With the above background in mind, the development of the solution algorithm and the computer program to solve problem (P) was undertaken and is described in Chapters 2 through 4.

## 1.4 Areas of Application

The solution algorithm and the computer program are designed to solve a multiactivity multifacility capacity-constrained 0-1 assignment problem, i.e., one which can be formulated as problem (P).

The basic elements of such a problem are activities that must be assigned, facilities and their meaningful configurations

represented as designs, the fixed and variable costs, and the capacity requirements of the activities.

The formulation (P) applies to existing and/or proposed facilities. In other words, it is useful for a situation where the decision may be to delete some of the existing facilities, as well as for a situation where the decision involves a selection out of a set of proposed facilities.

Table 1 includes examples of areas where formulation (P) is applicable. Within each application area, activities and facilities are identified. The implications of designs, variable costs, and fixed costs are apparent.

Obtaining the values of the data elements $b_k$, $d_{ijk}$, $s_k$, and in particular $a_{ij}$, can be a simple or a complex exercise depending on the particulars of the application, and the nature of the components comprising these elements. For example, in designing multi-echelon inventory systems [Gross, Pinkus, and Soland (1979)], $a_{ij}$ represents the inventory cost of product (activity) $j$ using echelon structure (design) $i$ and $b_k$ represents the fixed cost of installation (facility) $k$. The inventory cost $a_{ij}$ includes the cost of procurement, carrying inventory, filling orders, and stockouts. The value $a_{ij}$, and associated inventory stockage policies, are arrived at by solving a multi-echelon inventory problem. In other words, for product $j$ stocked under echelon structure $i$, optimal inventory policies are determined, at each installation of the structure, which yields $a_{ij}$. The facility fixed cost $b_k$ includes the capital expenditure for building the installation, along with a number of fixed costs associated with operating it, such as administrative expenses, the expense of renting the facility (if it is not built), and certain other fixed operating expenses.

In the case of designing a support system for repairable

TABLE 1

EXAMPLES OF APPLICATION AREAS

| Area of Application | Activities | Facilities |
|---|---|---|
| 1. Design of multi-echelon inventory systems* | Types of items to be stocked | Warehouses (Comprising different levels or echelons, e.g., central, regional, and local warehouses) |
| 2. Assignment of repairable components** | Major components of a unit, e.g., components of an aircraft, a ship, a piece of machinery | Repair depots |
| 3. Design of training programs | Training program categories or occupational classifications | Training schools |
| 4. Location of facilities*** | Types of services, e.g., health-care services | Buildings or installations, e.g., health-care centers |

*Gross, Pinkus, and Soland (1979)

**Gross and Pinkus (1979)

***Pinkus, Gross, and Soland (1973)

items [Gross and Pinkus (1979)], $a_{ij}$ represents the total variable cost if unit type (activity) $j$ is repaired under design $i$. The set of parameters taken into consideration to compute this cost for each unit type includes such things as varying population sizes, failure rates, average repair times, costs associated with their repair, the purchase and storage of spares, the purchase of repair channels, and travel to depots (facilities) for repair. A computer program is used to solve a spares and server provisioning problem, and the results provide the basic information to compute $a_{ij}$.

Thus, in general, the data elements of problem (P) may be obtained directly and/or by solving other related problem(s); it depends on the definition and the nature of the components comprising these data elements for a specific application area.

## 2. DEVELOPMENT OF THE SOLUTION ALGORITHM

The solution algorithm that has been developed to solve problem (P) is a branch-and-bound procedure which makes use of Lagrangian relaxation as a basic step.

This chapter considers two different Lagrangian relaxations of problem (P), their general characteristics, and some useful results leading to the specific case of Lagrangian relaxation utilized in the solution algorithm.

### 2.1 Lagrangian Relaxation

Taking a set of "complicating" constraints of a general mixed-integer program into the objective function in a Lagrangian fashion (with fixed multipliers) results in a "Lagrangian relaxation" of the original problem [Geoffrion (1974)]. The relaxed problem is easy to solve compared to the original problem, and provides a lower bound (for minimization problems) on the optimal value of the original problem.

Although the use of Lagrangian relaxation in discrete optimization has been reported prior to 1970 [e.g., Lorie and Savage (1955), Everett (1963), and Gilmore and Gomory (1963)], the "birth" of the Lagrangian approach as it exists today [Fisher (1978)] occurred in 1970 with the successful application of Lagrangian relaxations to the traveling salesman problem [Held and Karp (1970, 1971)]. This was followed by application of Lagrangian relaxation to scheduling problems [Fisher and Schrage (1972), and Fisher (1973, 1976)], the general integer programming problem [Shapiro (1971), and Fisher and Shapiro (1974)] and the generalized assignment problem [Ross and Soland (1975)]. Table 2 lists the applications of Lagrangian relaxation as given by Fisher (1978). A review of Lagrangian relaxation is also provided by Shapiro (1977) and Christofides (1980).

TABLE 2

APPLICATIONS OF LAGRANGIAN RELAXATION*

| Problem | Researchers | Lagrangian Problem |
|---|---|---|
| TRAVELING SALESMAN | | |
| Symmetric | Held & Karp (1970, 1971) | Spanning Tree |
| Asymmetric | Bazarra & Goode (1977) | Spanning Tree |
| Symmetric | Balas & Christofides (1976) | Perfect 2-Matching |
| Asymmetric | Balas & Christofides (1976) | Assignment |
| SCHEDULING | | |
| n\|m Weighted Tardiness | Fisher (1973) | Pseudo-Polynomial Dynamic Programming |
| 1 Machine Weight Tardiness | Fisher (1976) | Pseudo-Polynomial DP |
| Power Generation Systems | Muckstadt & Koenig (1977) | Pseudo-Polynomial DP |
| GENERAL IP | | |
| Unbounded Variables | Fisher & Shapiro (1974) | Group Problem |
| Unbounded Variables | Burdet & Johnson (1976) | Group Problem |
| 0 - 1 Variables | Etcheberry, et. al. (1978) | 0 - 1 GUB |
| LOCATION | | |
| Uncapacitated | Cornuejols, Fisher, & Nemhauser (1977) | 0 - 1 VUB |
| Capacitated | Geoffrion & McBride (1977) | 0 - 1 VUB |
| Databases in Computer Networks | Fisher & Hochbaum (1978) | 0 - 1 VUB |
| GENERALIZED ASSIGNMENT | | |
| | Ross & Soland (1975) | Knapsack |
| | Chalmet & Gelders (1976) | Knapsack, 0-1 GUB |
| SET COVERING--PARTITIONING | | |
| Covering | Etcheberry (1977) | 0 - 1 GUB |
| Partitioning | Nemhauser & Weber (1978) | Matching |

*Source: Fisher (1978)

### 2.1.1 Relaxing Problem (P)

By dividing constraints (5) by $s_k$ and letting $r_{ijk} = d_{ijk}/s_k$, problem (P) can be restated as follows.

$$
(P)\begin{cases}
\text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x_{ij} + \sum_{k=1}^{p} b_k y_k & (4)\\
\text{subject to} \quad \sum_{i=1}^{m} x_{ij} = 1 \qquad j=1,\dots,n & (2)\\
\sum_{i=1}^{m} \sum_{j=1}^{n} r_{ijk}\, x_{ij} \leq y_k \qquad k=1,\dots,p & (5')\\
x_{ij},\ y_k = 0 \text{ or } 1 \qquad \text{for all } i,j,k & (6)
\end{cases}
$$

A Lagrangian relaxation $(LR_u)$ of problem (P) relative to constraints (2) is obtained as

$$
(LR_u)\begin{cases}
\text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x_{ij} + \sum_{k=1}^{p} b_k y_k - \sum_{j=1}^{n} u_j \left( \sum_{i=1}^{m} x_{ij} - 1 \right) & (12)\\
\text{subject to} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ijk}\, x_{ij} \leq y_k \qquad k=1,\dots,p & (5')\\
x_{ij},\ y_k = 0 \text{ or } 1 \qquad \text{for all } i,j,k & (6)
\end{cases}
$$

where the $u_j$ are Lagrange multipliers; it follows that the optimal value of problem $(LR_u)$ is a lower bound on the optimal value of problem (P), i.e., $Z(LR_u) \leq Z(P)$. We will continue to use this notation in which $Z(\cdot)$ is the optimal value of problem $(\cdot)$.

Another Lagrangian relaxation $(LR_v)$ of problem (P), relative to constraints (5'), is obtained as

Minimize $\quad \sum_i \sum_j a_{ij} x_{ij} + \sum_k b_k y_k - \sum_k v_k \left( y_k - \sum_i \sum_j x_{ij} r_{ijk} \right)$

subject to (2) and (6), or equivalently,

$$(LR_v) \begin{cases} \text{Minimize} & \sum_i \sum_j x_{ij} \left( a_{ij} + \sum_k v_k r_{ijk} \right) - \sum_k y_k \left( v_k - b_k \right) & (13) \\ \text{subject to} & \sum_i x_{ij} = 1 \qquad j=1,\dots,n & (2) \\ & x_{ij},\ y_k = 0 \text{ or } 1 \qquad \text{for all } i,j,k & (6) \end{cases}$$

where the $v_k$ are non-negative Lagrange multipliers; it follows that $Z(LR_v) \leq Z(P)$ .

### 2.1.2 General Characteristics

A Lagrangian relaxation provides a lower bound on the optimal value of the orginal problem, i.e., in our case $Z(LR_u) \leq Z(P)$ and $Z(LR_v) \leq Z(P)$ . The usefulness of a Lagrangian relaxation depends on the closeness of this lower bound to the optimal value of the orginal problem. However, the relaxation must be "easy" to solve relative to the original problem. We observe that the optimal value of $y_k$ in problem $(LR_v)$ is 1 if $(v_k - b_k) \geq 0$ and 0 if $(v_k - b_k) \leq 0$ , and then problem $(LR_v)$ reduces to $n$ 0-1 "multiple choice" problems which are very easy to solve. On the other hand, problem $(LR_u)$ reduces to $k$ 0-1 knapsack problems. However, these problems are not independent because of the interaction of constraints (5') and the indirect relationship described earlier in Section 1.3 between the assignments and the facilities. In view of this complexity, relaxation $(LR_u)$ will not be considered further.

The choice of Lagrange multipliers in relaxation $(LR_v)$ should be such that $Z(LR_v)$ is as large as possible and hence as close as possible to $Z(P)$ in view of the relationship $Z(LR_v) \leq Z(P)$ . In other words, an equivalent problem is to find a vector v (representing $v_1$, $v_2$, $\dots, v_k$) to

$$(D)\quad \begin{cases} \text{Maximize} \quad \left[Z(LR_v)\right] & (14) \\ \qquad v \geq 0 \end{cases}$$

Obviously, $Z(LR_v) \leq Z(D) \leq Z(P)$ .

The general properties of Lagrangian relaxation have been well described in the literature [e.g., Geoffrion (1974), Geoffrion and McBride (1978), and Fisher (1980)]. Some of these properties relating the Lagrangian relaxation and the usual LP relaxation are stated below.

The LP relaxation $(\bar{P})$ of problem (P) is obtained by relaxing the integrality constraints (6), i.e., the formulation $(\bar{P})$ is

$$(\bar{P})\quad \begin{cases} \text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x_{ij} + \sum_{k=1}^{p} b_k y_k & (4) \\ \text{subject to} \quad \sum_{i=1}^{m} x_{ij} = 1 \qquad j=1,\ldots,n & (2) \\ \qquad \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ijk}\, x_{ij} \leq y_k \qquad k=1,\ldots,p & (5') \\ \qquad y_k \leq 1 \qquad k=1,\ldots,p & (15) \\ \qquad x_{ij},\ y_k \geq 0 \qquad \text{for all } i,j,k & (16) \end{cases}$$

Note that the constraints $x_{ij} \leq 1$ are implicit in constraints (2).

Also consider the following partial convex hull relaxation $(P^*)$ of problem (P).

$$(P^*)\quad \begin{cases} \text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x_{ij} + \sum_{k=1}^{p} b_k y_k & (4) \\ \text{subject to} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ijk}\, x_{ij} \leq y_k \qquad k=1,\ldots,p & (5') \\ \qquad x_{ij},\ y_k \in \text{convex hull } \{(2),\ (6)\} & (17) \end{cases}$$

Then the relationships between the optimal values of various problems [Geoffrion and McBride (1978)] are as follows.

$$Z(\bar{P}) \leq Z(LR_{\tilde{v}}) \leq \max_{v \geq 0} Z(LR_v) = Z(D) = Z(P^*) \leq Z(P) \tag{18}$$

where $\tilde{v}$ are the values $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_p$ of a dual optimal solution of $(\bar{P})$ corresponding to constraints (5').

Thus, the optimal dual solution associated with the usual LP relaxation furnishes a choice of Lagrange multipliers such that the associated Lagrangian relaxation is at least as tight as the usual LP relaxation, and generally a good deal tighter and even as tight as the partial convex hull relaxation.

Since $Z(D) = Z(P^*)$ , the quality of the bound obtained from the Lagrangian relaxation depends on where $Z(P^*)$ lies in the range between $Z(\bar{P})$ and $Z(P)$ . It turns out that problem $(LR_v)$ possesses the "integrality property," i.e., the optimal value of problem $(LR_v)$ is not altered by dropping the integrality conditions on its variables and therefore [Geoffrion (1974)]

$$Z(D) = Z(P^*) = Z(\bar{P}) \tag{19}$$

Thus, the Lagrangian relaxation $(LR_v)$ is no better than the LP relaxation $(\bar{P})$. On the other hand, Lagrangian relaxation $(LR_u)$ does not possess the integrality property and, hence, could provide an equal or better bound than the LP relaxation $(\bar{P})$ ; but the computational difficulties do not favor pursuing formulation $(LR_u)$ .

It is possible to consider alternative formulations of problem (P) with the objective of obtaining tighter bounds. This aspect is discussed in Chapter 6.

## 2.2 Some Results

We now turn to the basic question of choosing Lagrange multipliers v so that $(LR_v)$ is optimal to the extent possible, which is equivalent to solving problem (D). We also need to consider this question when some of the $x_{ij}$ and $y_k$ variables have been assigned values of 1 or 0, i.e., at a node other than the starting or "root" node in the branch-and-bound tree. For this purpose, some terminology is defined and formulations corresponding to problems (P), $(LR_v)$ and (D) are first developed. Then some important results pertaining to the choice of Lagrange multipliers will be proved. Gavish (1978) provides a method of obtaining the 'best' multipliers, based on solving an equivalent linear programming problem. Such a formulation is difficult in our case, and, besides, we propose to avoid solving LP problems in our branch-and-bound procedure.

Define the sets

$S = \{(i,j) | x_{ij}$ has an assigned value of 1 or 0$\}$ , and

$T = \{k | y_k$ has an assigned value of 1 or 0$\}$ .

These sets represent the partial solution of problem (P) and the variables contained in these sets are termed fixed variables. [Geoffrion (1967)]. Let $\bar{S}$ and $\bar{T}$ represent the corresponding complementary sets, i.e., comprised of the $x_{ij}$ and $y_k$ variables, which have not been assigned specific values and, therefore, are called free variables. A completion of a partial solution is defined as a solution that is determined by S and T together with a binary specification (0 or 1) of the values of the free $x_{ij}$ and $y_k$ variables from sets $\bar{S}$ and $\bar{T}$ .

Let $S \cup \bar{S} = S_1$ and $T \cup \bar{T} = T_1$ .

Consider a partial solution to problem (P) in which specific values (of 1 or 0) are assigned to some of the $x_{ij}$ and $y_k$ such that

$$\sum_{\substack{i=1 \\ (i,j)\varepsilon S}}^{m} x_{ij} \leq 1 \qquad \forall j \ ,$$

and

$$\sum_{\substack{i \ j \\ (i,j)\varepsilon S}} \sum r_{ijk} \, x_{ij} \leq y_k \qquad \forall k\varepsilon T$$

$$\sum_{\substack{i \ j \\ (i,j)\varepsilon \bar{S}}} \sum r_{ijk} \, x_{ij} \leq 1 \qquad \forall k\varepsilon \bar{T}$$

and such that $x_{ij} = 1$ and $e_{ik} = 1$ imply that $k\varepsilon T$ and $y_k = 1$. Recall that, by definition, $e_{ik} = 1$ if design $i$ uses facility $k$, and $e_{ik} = 0$ otherwise.

The problem of finding an optimal completion of the partial solution of problem (P) can be stated as follows.

$$(P_\ell) \left\{ \begin{array}{lll} \text{Minimize} & \displaystyle\sum_{\substack{i \ j \\ (i,j)\varepsilon\bar{S}}}\sum a_{ij} x_{ij} + \sum_{\substack{k \\ k\varepsilon\bar{T}}} b_k y_k + \sum_{\substack{i \ j \\ (i,j)\varepsilon S}}\sum a_{ij} x_{ij} + \sum_{\substack{k \\ k\varepsilon T}} b_k y_k & (20) \\ \text{subject to} & \displaystyle\sum_{\substack{i \\ (i,j)\varepsilon\bar{S}}} x_{ij} = 1 - \sum_{\substack{i \\ (i,j)\varepsilon S}} x_{ij} \qquad \forall j & (21) \\ & \displaystyle\sum_{\substack{i \ j \\ (i,j)\varepsilon\bar{S}}}\sum r_{ijk} x_{ij} \leq y_k - \sum_{\substack{i \ j \\ (i,j)\varepsilon S}}\sum r_{ijk} x_{ij} \qquad \forall k & (22) \\ & x_{ij}, \, y_k = 0 \text{ or } 1 \qquad \forall (i,j)\varepsilon\bar{S} \ , \ k\varepsilon\bar{T} & (23) \end{array} \right.$$

We call this problem $(P_\ell)$ where $\ell$ indicates the node in the branch-and-bound tree.

A Lagrangian relaxation of problem $(P_\ell)$ with respect to constraints (22) is obtained by introducing non-negative Lagrange multipliers $v_k$, $k=1,2,\ldots,p$ ; the relaxation is then

$$\text{Minimize} \quad \sum_{\substack{i\ j \\ (i,j)\varepsilon\bar{S}}} a_{ij}\, x_{ij} + \sum_{\substack{k \\ k\varepsilon\bar{T}}} b_k y_k + \sum_{\substack{i\ j \\ (i,j)\varepsilon S}} a_{ij}\, x_{ij} + \sum_{\substack{k \\ k\varepsilon T}} b_k y_k$$

$$- \sum_{k} v_k \left[ y_k - \sum_{\substack{i\ j \\ (i,j)\varepsilon\bar{S}}} r_{ijk}\, x_{ij} - \sum_{\substack{i\ j \\ (i,j)\varepsilon S}} r_{ijk}\, x_{ij} \right] \qquad (24)$$

$$\text{subject to} \quad \sum_{\substack{i \\ (i,j)\varepsilon\bar{S}}} x_{ij} = 1 - \sum_{\substack{i \\ (i,j)\varepsilon S}} x_{ij} \qquad \forall j \qquad (21)$$

$$x_{ij},\ y_k = 0 \text{ or } 1 \qquad \forall (i,j)\varepsilon\bar{S}\ ,\ k\varepsilon\bar{T} \qquad (23)$$

Rearranging (24), and using the relationship $T_1 = T \cup \bar{T}$, we have problem $(LR_{\ell,v})$ :

$$(LR_{\ell,v}) \begin{cases} \text{Minimize} \quad \sum\limits_{\substack{i\ j \\ (i,j)\varepsilon\bar{S}}} x_{ij} \left( a_{ij} + \sum\limits_{k\varepsilon T_1} v_k\, r_{ijk} \right) + \sum\limits_{\substack{i\ j \\ (i,j)\varepsilon S}} x_{ij} \left( a_{ij} + \sum\limits_{k\varepsilon T_1} v_k\, r_{ijk} \right) & \\ \qquad - \sum\limits_{k\varepsilon\bar{T}} y_k \left( v_k - b_k \right) - \sum\limits_{k\varepsilon T} y_k \left( v_k - b_k \right) & (25) \\ \text{subject to} \quad \sum\limits_{\substack{i \\ (i,j)\varepsilon\bar{S}}} x_{ij} = 1 - \sum\limits_{\substack{i \\ (i,j)\varepsilon S}} x_{ij} \qquad \forall j & (21) \\ \qquad x_{ij},\ y_k = 0 \text{ or } 1 \qquad \forall (i,j)\varepsilon\bar{S}\ ,\ k\varepsilon\bar{T} & (23) \end{cases}$$

Then we have $Z(LR_{\ell,v}) \leq Z(P_\ell)$ . An important problem is the choice of Lagrange multipliers $v_1, v_2, \ldots, v_p$ , represented by vector $v$ , that maximize $Z(LR_{\ell,v})$ , i.e., the problem $(D_\ell)$ :

$$(D_\ell) \begin{cases} \text{Maximize} \left[ Z(LR_{\ell,v}) \right] & (26) \\ v \geq 0 & \end{cases}$$

We now state and prove some theorems related to the choice of Lagrange multipliers $v_1, v_2, \ldots, v_p$ .

<u>Theorem 1</u>: There exists an optimal solution to problem (D) in which $v_k \geq b_k$ for all $k$ .

<u>Proof</u>: Suppose $v_1 < b_1$ , in an optimal solution to problem (D), i.e., $Z(D) = Z(LR_{v*})$ where $v_1^* < b_1$ .

Recall that

$$Z(LR_{v*}) = \text{Min} \sum_i \sum_j x_{ij} \left(a_{ij} + \sum_k v_k^* r_{ijk}\right) - \sum_k y_k \left(v_k^* - b_k\right)$$

$$\text{s.t.} \quad \sum_i x_{ij} = 1 \qquad \forall j \qquad (2)$$

$$x_{ij}, y_k = 0 \text{ or } 1 \qquad \forall i,j,k \qquad (6)$$

For $v_1^* < b_1$ , the optimal value of $y_1$ is $0$ , and the term $- y_1 (v_1^* - b_1)$ in the objective function is 0.

Consider what happens if we increase $v_1^*$ to $b_1$ . Call the resulting vector $\underline{v}$ . Consider problem $(LR_{\underline{v}})$ . The optimal value of $y_1$ in problem $(LR_{\underline{v}})$ is $0$ or $1$ , and the term $- y_1 \ (\underline{v}_1 - b_1)$ is $0$ . However, the optimal value of $y_k$ is the same in problems $(LR_{v^*})$ and $(LR_{\underline{v}})$ for all $k > 1$ . Therefore, the quantity $\sum_k y_k (v_k - b_k)$ is the same at the optimal solution for both $v = v^*$ and $v = \underline{v}$ .

Since $\underline{v}_1 > v_1^*$ , we note that in the objective function,

$$a_{ij} + \sum_{k=1}^{p} \underline{v}_k r_{ijk} \geq a_{ij} + \sum_{k=1}^{p} v_k^* r_{ijk} \qquad \forall i,j ,$$

and therefore $Z(LR_{\underline{v}}) \geq Z(LR_{v^*})$ .

It follows that there is an optimal solution to problem (D) in which $v_1 \geq b_1$ .

Since the choice of $k=1$ was arbitrary, the same result holds for any value of $k$, $k=1,\ldots,p$ ; hence, there exists an optimal solution to problem (D) in which $v_k \geq b_k$ for all $k$ .

<u>Theorem 2</u>: There exists an optimal solution to problem $(D_\ell)$ in which $v_k \geq b_k$ if (i) $k\varepsilon\bar{T}$ or (ii) $k\varepsilon T$ and $y_k = 0$ .

<u>Proof</u>: Suppose $v_1 < b_1$ in an optimal solution to problem $(D_\ell)$ , i.e., $Z(D_\ell) = Z(LR_{\ell,v^*})$ where $v_1^* < b_1$ . Then $k=1$ can be such that $k\varepsilon T$ or $k\varepsilon\bar{T}$ .

<u>Case (i)</u>: Let $k\varepsilon\bar{T}$ .

Recall that

$$
\begin{aligned}
Z(LR_{\ell,v^*}) = \text{Min} & \sum_{\substack{i \\ (i,j)\varepsilon\bar{S}}}\sum_{j} x_{ij}\left(a_{ij} + \sum_{k} v_k^* r_{ijk}\right) \\
& + \sum_{\substack{i \\ (i,j)\varepsilon S}}\sum_{j} x_{ij}\left(a_{ij} + \sum_{k} v_k^* r_{ijk}\right) \\
& - \sum_{k\varepsilon\bar{T}} y_k\left(v_k^* - b_k\right) - \sum_{k\varepsilon T} y_k\left(v_k^* - b_k\right)
\end{aligned}
$$

$$
\text{s.t.} \quad \sum_{\substack{i \\ (i,j)\varepsilon\bar{S}}} x_{ij} = 1 - \sum_{\substack{i \\ (i,j)\varepsilon S}} x_{ij} \qquad \forall j \tag{21}
$$

$$
x_{ij},\ y_k = 0 \text{ or } 1 \qquad \forall (i,j)\varepsilon\bar{S}\ ,\ k\varepsilon\bar{T} \tag{23}
$$

For $v_1^* < b_1$ , and $k\varepsilon\bar{T}$ , the optimal value of $y_1$ is 0 and the term $- y_1\,(v_1^* - b_1)$ in the objective function is 0 .

Let $v_1^*$ be increased to $b_1$ ; call the resulting vector $\underline{v}$ . Consider problem $(LR_{\ell,\underline{v}})$ . The optimal value of $y_1$ in $(LR_{\ell,\underline{v}})$ is 0 or 1 , then the term $-y_1(\underline{v}_1 - b_1)$ is 0 . For $k > 1$ , the optimal value of $y_k$ being the same in $(LR_{\ell,v^*})$ and $(LR_{\ell,\underline{v}})$ , we find that $\sum_{k\varepsilon T_1} y_k (v_k - b_k)$ is the same at the optimal solution for both $v = v^*$ and $v = \underline{v}$ . But $\underline{v}_1 > v_1^*$ ; therefore

$$a_{ij} + \sum_{k\varepsilon T_1} \underline{v}_k r_{ijk} \geq a_{ij} + \sum_{k\varepsilon T_1} v_k^* r_{ijk} \quad \forall (i,j)\varepsilon S \text{ and } (i,j)\varepsilon \bar{S}$$

Hence, $Z(LR_{\ell,\underline{v}}) \geq Z(LR_{\ell,v^*})$ , wherefrom it follows that there is an optimal solution to $(D_\ell)$ in which $v_1 \geq b_1$ . Since the choice of $k=1$ was arbitrary, the same results hold for any value of $k$, $k\varepsilon\bar{T}$ . Hence, there exists an optimal solution to problem $(D_\ell)$ in which $v_k \geq b_k$ for all $k\varepsilon\bar{T}$ .

<u>Case (ii)</u>: Let $k\varepsilon T$ and $y_k = 0$

Considering problem $(LR_{\ell,v^*})$ , for $k = 1$ , $v_1^* < b_1$ , and $y_1 = 0$ , the term $-y_1(v_1^* - b_1)$ in the objective function is 0 .

Increase $v_1^*$ to $b_1$ and call the resulting vector $\underline{v}$ . The term $-y_1(\underline{v}_1 - b_1)$ is 0 . For $k > 1$ , the optimal values of $y_k$ are the same in problems $(LR_{\ell,v^*})$ and $(LR_{\ell,\underline{v}})$ . Therefore $\sum_{k\varepsilon T_1} y_k(v_k - b_k)$ is the same at the optimal solution for both $v = v^*$ and $v = \underline{v}$ . Since $\underline{v}_1 > v_1^*$ ,

$$a_{ij} + \sum_{k\varepsilon T_1} \underline{v}_k r_{ijk} \geq a_{ij} + \sum_{k\varepsilon T_1} v_k^* r_{ijk} \quad \forall (i,j)\varepsilon S \text{ and } (i,j)\varepsilon \bar{S}$$

Therefore $Z(LR_{\ell,\underline{v}}) \geq Z(LR_{\ell,v^*})$ . It follows that there exists an optimal solution to $(D_\ell)$ in which $v_1 \geq b_1$ . The choice of $k=1$ being arbitrary, the same results hold for any value of $k$ , $k\varepsilon T$ and $y_k = 0$ ; which proves case (ii) of the Theorem.

It may be added that there is another possibility which complements case (ii) of Theorem 2, i.e., if $k\varepsilon T$ and $y_k = 1$ . We treat this possibility as a conjecture since a result similar to the one above could not be proved, as discussed now.

With $y_1 = 1$ and $v_1^* < b_1$ , we observe from problem $(LR_{\ell,v^*})$ that for a solution vector $X^*$ (with elements $x_{ij}^*$) and $Y^*$ (with elements $y_1^*,\ldots,y_p^* | y_1^* = 1$ and $y_2^*,\ldots,y_p^* = 0$ or 1) ,

$$
\begin{aligned}
Z(LR_{\ell,v^*}) = & \sum_{\substack{i\ j \\ (i,j)\varepsilon\bar{S}}} x_{ij}^* \left(a_{ij} + v_1^* r_{ij1} + \sum_{k>1} v_k^* r_{ijk}\right) \\
& + \sum_{\substack{i\ j \\ (i,j)\varepsilon S}} x_{ij}^* \left(a_{ij} + v_1^* r_{ij1} + \sum_{k>1} v_k^* r_{ijk}\right) \\
& - \sum_{k\varepsilon\bar{T}} y_k^* \left(v_k^* - b_k\right) - y_1\left(v_1^* - b_1\right) - \sum_{\substack{k>1 \\ k\varepsilon T}} y_k^* \left(v_k^* - b_k\right)
\end{aligned}
$$

Since $v_1^* < b_1$ and $y_1 = 1$ the term $- y_1(v_1^* - b_1)$ is positive. If we raise $v_1^*$ to $b_1$ , say $\underline{v}_1$ , the term $- y_1(\underline{v}_1 - b_1)$ is 0 .

The difference between $Z(LR_{\ell,v^*})$ and the objective function value of problem $(LR_{\ell,\underline{v}})$ with $X=X^*$ and $Y=Y^*$ is

$$
\begin{aligned}
& = \sum_i \sum_j x_{ij}^* v_1^* r_{ij1} + (b_1 - v_1^*) - \sum_i \sum_j x_{ij}^* b_1 r_{ij1} \\
& = (b_1 - v_1^*) - \sum_i \sum_j x_{ij}^* (b_1 - v_1^*) r_{ij1} .
\end{aligned}
$$

This difference can be either negative or positive, and so we cannot conclude that there is an optimal solution to problem $(D_\ell)$ in which

$v_1 \geq b_1$ . We believe this conclusion to be false.

Theorem 3: Let $(X^*, Y^*)$ solve problem $(LR_v)$ for $v_k = b_k$ for all $k$ . If $(X^*, Y^*)$ is feasible for problem (P), there exists an optimal solution to problem (D) in which $v_k = b_k$ for all $k$ .

Proof: In view of Theorem 1, there exists an optimal solution to (D) in which $v_k \geq b_k$ for all $k$ , i.e., $v \geq b$ . Let $\underline{v}$ be such an optimal $v$ . We will show that $Z(LR_{\underline{v}}) \leq Z(LR_b)$ , from which it follows that $v = b$ solves problem (D) .

Recall that

$$Z(LR_{\underline{v}}) = \min_{X,Y} \sum_i \sum_j x_{ij} \left(a_{ij} + \sum_k \underline{v}_k r_{ijk}\right) - \sum_k y_k \left(\underline{v}_k - b_k\right)$$

$$\text{s.t.} \quad \sum_i x_{ij} = 1 \qquad \forall j \qquad (2)$$

$$x_{ij}, y_k = 0 \text{ or } 1 \qquad \forall i,j,k \qquad (6)$$

Since $\underline{v} \geq b$ , $y_k = 1 \ \forall k$ is an optimal choice.

Hence,
$$Z(LR_{\underline{v}}) = \min_{X} \sum_i \sum_j x_{ij} \left(a_{ij} + \sum_k \underline{v}_k r_{ijk}\right) - \sum_k \left(\underline{v}_k - b_k\right)$$

$$\text{s.t.} \quad \sum_i x_{ij} = 1 \qquad \forall j \qquad (2)$$

$$x_{ij} = 0 \text{ or } 1 \qquad \forall i,j \qquad (6a)$$

Now consider $(LR_b)$ . Since $v = b$ , the last term of the objective function drops out, and we have

$$Z(LR_b) = \min_{X,Y} \sum_i \sum_j x_{ij} \left(a_{ij} + \sum_k b_k r_{ijk}\right)$$

subject to (2) and (6)

$$= \min_X \sum_i \sum_j x_{ij} \left( a_{ij} + \sum_k b_k r_{ijk} \right)$$

subject to (2) and (6a)

$$= \sum_i \sum_j x_{ij}^* \left( a_{ij} + \sum_k b_k r_{ijk} \right)$$

where $X^*$ with elements $x_{ij}^*$ is the minimizing solution vector which satisfies (2) and (6a) .

Now $(X^*, Y^*)$ feasible for (P) implies that

$$\sum_i \sum_j x_{ij}^* r_{ijk} \leq y_k^* \leq 1 \qquad \forall k .$$

Hence, $\sum_k \left( \underline{v}_k - b_k \right) \sum_i \sum_j x_{ij}^* r_{ijk} \leq \sum_k \left( \underline{v}_k - b_k \right)$ ,

or $$\sum_k \left( \underline{v}_k - b_k \right) \sum_i \sum_j x_{ij}^* r_{ijk} - \sum_k \left( \underline{v}_k - b_k \right) \leq 0 \qquad (27)$$

Rewriting, $$Z(LR_{\underline{v}}) = \min_X \sum_i \sum_j x_{ij} \left[ a_{ij} + \sum_k r_{ijk} \left( b_k + \left( \underline{v}_k - b_k \right) \right) \right] - \sum_k \left( \underline{v}_k - b_k \right)$$

subject to (2) and (6a)

$$= \min_X \left\{ \sum_i \sum_j x_{ij} \left( a_{ij} + \sum_k r_{ijk} b_k \right) + \sum_k \left( \underline{v}_k - b_k \right) \sum_i \sum_j x_{ij} r_{ijk} - \sum_k \left( \underline{v}_k - b_k \right) \right\}$$

subject to (2) and (6a)

$$= \sum_i \sum_j x_{ij}^* \left( a_{ij} + \sum_k r_{ijk} b_k \right) + \sum_k \left( \underline{v}_k - b_k \right) \sum_i \sum_j x_{ij}^* r_{ijk} - \sum_k \left( \underline{v}_k - b_k \right)$$

$$\leq \sum_i \sum_j x_{ij}^* \left( a_{ij} + \sum_k r_{ijk} b_k \right) = Z(LR_b)$$

by (27), or $Z(LR_v) \leq Z(LR_b)$ ; it follows that $v = b$ solves problem (D).

## 2.3 Relaxation ($PR_\ell$)

Theorems 1 and 3 are useful in providing a choice of Lagrange multipliers as a starting point in solving a relaxation of problem (P) at the root node. Theorem 2, similar to Theorem 1, provides results for a partial solution of problem (P), i.e., at a node other than the root node where some of the $x_{ij}$ and $y_k$ have been fixed at 1 or 0 .

Theorem 1 is important in pointing out that a certain set of Lagrange multipliers $v$ such that $v_k \geq b_k$ for all $k$ would provide an optimal choice. Theorem 3 narrows this choice to $v_k = b_k$ for all $k$ for a specific situation, i.e., when the resulting solution is feasible for problem (P).

Letting $v_k = b_k$ for all $k$ , problem $(LR_v)$ becomes:

$$(LR_b) \begin{cases} \text{Minimize} \quad \sum_i \sum_j c_{ij} x_{ij} & (28) \\ \text{subject to} \quad \sum_i x_{ij} = 1 \quad \forall j & (2) \\ \qquad x_{ij} = 0 \text{ or } 1 \quad \forall i,j & (6a) \end{cases}$$

$$\text{where} \quad c_{ij} = a_{ij} + \sum_k b_k r_{ijk} . \qquad (29)$$

Note that problem $(LR_b)$ is very easy to solve; its optimal value is just the sum of the minimum (over i) $c_{ij}$ for all $j$ , i.e.,

$$Z(LR_b) = \sum_j \min_i \{c_{ij}\} \qquad (30)$$

We solve this problem as a starting point at the root node in our branch-and-bound procedure.

As we move to other nodes by fixing some of the variables, we must deal with problems having the form of problem $(P_\ell)$ instead of problem (P). The appropriate relaxation is then problem $(LR_{\ell,v})$, whose optimal value $Z(LR_{\ell,v})$ is the lower bound required at node $\ell$. Our algorithm branches only on $x_{ij}$ variables and uses the constraints (5') to fix appropriate $y_k$ variables at values of 1. More precisely, if $x_{ij}$ is fixed at 1 and $e_{ik} = 1$, then $y_k$ must be 1 in every feasible completion of problem (P) so we can include the index $k$ in $T$ and fix $y_k$ at 1. To account for the various possible combinations of $i$ and $j$, we define

$$\alpha_{k\ell} = 1 \text{ if } x_{ij}\, e_{ik} > 0 \text{ for any } (i,j)\in S ,$$
$$= 0 \text{ otherwise .} \tag{31}$$

At any node $\ell$ then, $y_k$ is fixed at 1 and $k\in T$ if $\alpha_{k\ell} = 1$.

There is another way in which it is appropriate to fix $y_k$ at 1 at node $\ell$. If the available choice of designs for some activity $j$ requires the use of facility $k$, then $y_k$ may be set to 1. Formally, define

$$W = \{j \mid (i,j)\in S \text{ and } x_{ij} = 1 \text{ for some } i\} \tag{32}$$

and its complement $\bar{W}$. Then define

$$\beta_{k\ell} = 1 \text{ if } \sum_{j\in\bar{W}} \min_{\substack{i \\ (i,j)\in\bar{S}}} d_{ijk} > 0 ,$$
$$= 0 \text{ otherwise .} \tag{33}$$

Then $y_k$ is fixed at 1 and $k\in T$ if $\beta_{k\ell} = 1$. It is convenient to combine these two notations in forcing $y_k$ to 1. Define

$$\delta_{k\ell} = \text{Max}\{\alpha_{k\ell}\ ,\ \beta_{k\ell}\} \tag{34}$$

so $y_k$ is fixed at 1 and $k\varepsilon T$ if $\delta_{k\ell} = 1$ .

To return to the relaxation, problem $(LR_{\ell,v})$, we must make a choice of the vector $v$ of Lagrange multipliers. Of course, we would like to use an optimal choice, i.e., a vector $v$ that solves problem $(D_\ell)$ . Recall, however, that Theorem 2 did not provide us any useful information about the optimal value of $v_k$ if $k\varepsilon T$ and $y_k = 1$ . To simplify our approach and have recourse to the results of Theorems 1 and 3, we choose $v_k = 0$ if $k\varepsilon T$ and $y_k = 1$ . Note that there are no $k\varepsilon T$ such that $y_k = 0$ because of practical considerations and because our branching rule only results in fixing $y_k$ values at 1 . Problem $(LR_{\ell,v})$ now takes the form

$$(LR_{\ell,\bar{v}})\begin{cases}\text{Minimize} & \sum_i \sum_j x_{ij}\left(a_{ij} + \sum_{k\varepsilon\bar{T}} v_k r_{ijk}\right) - \sum_{k\varepsilon\bar{T}} y_k\left(v_k - b_k\right) + \sum_{k\varepsilon T} b_k \quad (35)\\ \text{subject to} & \sum_i x_{ij} = 1\ , \qquad \forall j \quad (6)\\ & x_{ij},\ y_k = 0 \text{ or } 1 \text{ for all } (i,j)\varepsilon\bar{S}\ ,\ k\varepsilon T\ . \quad (23)\end{cases}$$

Note that in this problem $(LR_{\ell,\bar{v}})$, $\bar{v}_k = 0$ if $k\varepsilon T$ . Also note how closely it resembles problem $(LR_v)$, the relaxation at the root node. As in that case, we would like the lower bound $Z(LR_{\ell,\bar{v}})$ to be as large as possible, i.e., we seek $\bar{v}$ to

$$(D_{\bar{\ell}})\begin{cases}\text{Maximize} & \left[Z(LR_{\ell,\bar{v}})\right] \quad (36)\\ & \bar{v} \geq 0\end{cases}$$

Because of the close similarity of problems $(LR_{\ell,\bar{v}})$ and $(LR_v)$, it is possible to obtain results about problem $(D_{\bar{\ell}})$ that are analogous to those obtained about problem (D). We state these results as Theorems 4 and 5. Their proofs are omitted because they follow precisely the

proofs of Theorems 1 and 3, respectively, and their validity follows from the fact that problem $(LR_{\ell,\bar{v}})$ is essentially the same as problem $(LR_v)$ but involves only the free variables.

Theorem 4: There exists an optimal solution to problem $(D_{\bar{\ell}})$ in which $v_k \geq b_k$ for all $k \epsilon \bar{T}$ .

Theorem 5: Let $(X^*, Y^*)$ solve problem $(LR_{\ell,\bar{v}})$ for $v_k = b_k$ for all $k \epsilon \bar{T}$ . If $(X^*, Y^*)$ satisfies (5') for all $k \epsilon \bar{T}$ , there exists an optimal solution to problem $(D_{\bar{\ell}})$ in which $v_k = b_k$ for all $k \epsilon \bar{T}$ .

Just as Theorems 1 and 3 motivated us to use the relaxation problem $(LR_b)$ to obtain our lower bound at node 1 , Theorems 4 and 5 motivate us to set $v_k = b_k$ for all $k \epsilon \bar{T}$ in relaxation problem $(LR_{\ell,\bar{v}})$ to obtain our lower bound at node $\ell$ . With this specification, problem $(LR_{\ell,\bar{v}})$ becomes

$$(PR_\ell) \begin{cases} \text{Minimize} & \sum_i \sum_j c_{ij\ell}\, x_{ij} + FC_\ell & (37) \\ \text{subject to} & \sum_i x_{ij} = 1 \qquad \forall j & (2) \\ & x_{ij} = 0 \text{ or } 1 \qquad \forall (i,j) \epsilon \bar{S} \ , & (23a) \end{cases}$$

where

$$c_{ij\ell} = a_{ij} + \sum_{k \epsilon \bar{T}} b_k\, r_{ijk}$$

$$= a_{ij} + \sum_{k=1}^{p} b_k \left(1 - \delta_{k\ell}\right) r_{ijk} \tag{38}$$

and the fixed cost $FC_\ell$ is given by

$$FC_\ell = \sum_{k \epsilon T} b_k = \sum_{k=1}^{p} \delta_{k\ell}\, b_k \ . \tag{39}$$

This specific relaxation, problem $(PR_\ell)$, is of the same form as problem $(LR_b)$ and is equally easy to solve in one pass. Its optimal value $Z(PR_\ell)$ serves as the lower bound at node $\ell$ . Note that for $\ell=1$ , problem $(PR_\ell)$ is the same as problem $(LR_b)$ .

It is clear that setting each Lagrange multiplier $v_k$ to $b_k$ for $k\epsilon\bar{T}$ and to 0 for $k\epsilon T$ is not generally optimal in terms of achieving the tightest lower bound (except as per Theorem 3). But it provides a good starting point in seeking an optimal vector $v$ and it provides an easily calculated lower bound at each node of our branch-and-bound procedure. The question of how to improve upon this choice of multiplier values will be discussed in Chapter 6.

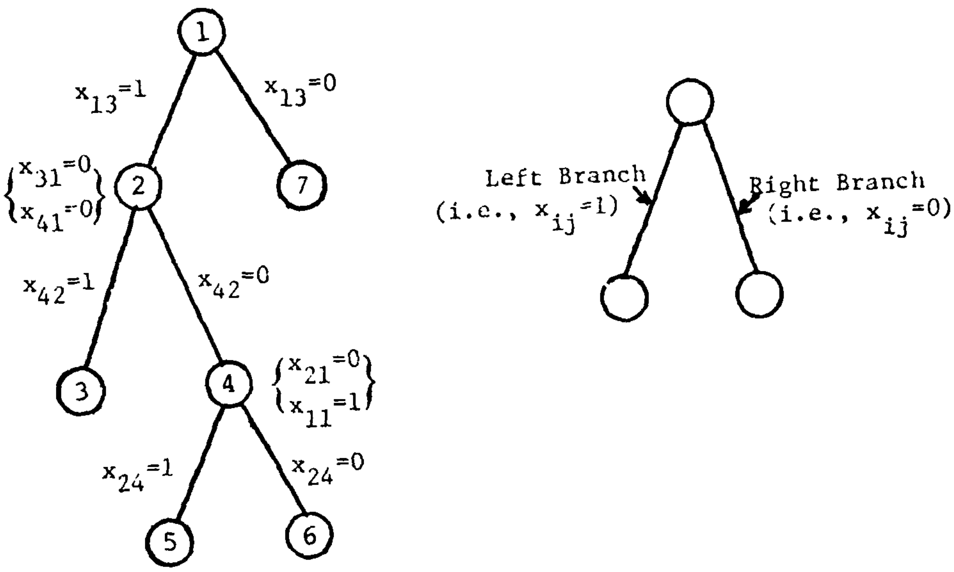
## 3. METHODOLOGY FRAMEWORK

The branch-and-bound procedure/methodology developed to solve problem (P) uses Lagrangian relaxation ($PR_\ell$) as a basic step. The branching rule dictates which $x_{ij}$ variable to branch on at each node. In addition, there are certain rules (e.g., the capacity rule and the bounding rule) which contribute, significantly, in improving the overall efficiency of the procedure.

Some basic terms such as fixed and free variables, partial solution and its completion were introduced in the previous chapter. This chapter first provides a preliminary discussion of the branch-and-bound methodology, [Geoffrion (1967), and Geoffrion and Marsten (1972)]. Representation and storage of the $x_{ij}$ variables for branching and backtracking is described in order to provide continuity and consistency with the computer program covered in Chapter 4. This is followed by a description of the major components of the branch-and-bound methodology.

Branching and backtracking is done on the $x_{ij}$ variables. The branching commences by fixing the $x_{ij}$ variable (selected by the branching rule) to 1 and moving to the left branch node. When backtracking, we fix the corresponding $x_{ij}$ variable at 0 and move to the right branch node (if the right branch node has not already been explored). An $x_{ij}$ variable can also be fixed at 0 or 1 by rules other than the branching rule. The capacity rule and the bounding rule are two such rules employed in our methodology.

Figure 4a shows a branch-and-bound tree. The $x_{ij}$ variables fixed at 0 or 1 at any node due to rules other than the branching rule are shown in parenthesis at the appropriate node.




Node (1) is the root node and also the parent node for nodes (2) and (7)

Node (2) is the parent node for nodes (3) and (4), etc.

Figure 4a.

A branch-and-bound tree illustration

| Node ($\ell$) | Partial Solution ($S_\ell$) |
|---|---|
| 1 | $\phi$ |
| 2 | {103, - 301, - 401} |
| 3 | {103, - 301, - 401, 402} |
| 4 | {103, - 301, - 401, - 402, - 201, 101} |
| 5 | {103, - 301, - 401, - 402, - 201, 101, 204} |
| 6 | {103, - 301, - 401, - 402, - 201, 101, -204} |
| 7 | {- 103} |

Figure 4b.

Partial solutions for the above illustration (Figure 4a)

For problem $(PR_\ell)$ , a partial solution corresponding to set $S$ at node $\ell$ , i.e , $S_\ell$ contains $x_{ij}$ variables assigned values of 1 or 0 . For simplicity in the computer program, an $x_{ij}$ variable fixed at 1 is represented as $(100\ i + j)$ , whereas an $x_{ij}$ variable fixed at 0 as $-(100\ i + j)$ , e.g., $x_{32} = 1$ and $x_{32} = 0$ are represented as 302 and - 302 respectively. Since branching is done on $x_{ij}$ variables, it is necessary to make a distinction between $x_{ij}$ variables fixed at 1 due to the branching rule and those fixed at 1 due to the other rules. We make this distinction by underlining the positive number to represent an $x_{ij}$ fixed at 1 due to the other rules. For example, 204, - 301, 103 represent, respectively, $x_{24} = 1$ due to the branching rule, $x_{31} = 0$ due to the branching rule or any other rule, and $x_{13} = 1$ due to a rule other than the branching rule.

Figure 4b shows the partial solutions $S_\ell$ of the branch-and-bound tree in Figure 4a.

Implicit enumeration involves generating a sequence of partial solutions and simultaneously considering all completions of each. For our minimization problem, we start with an initial solution having a very large value (infinity) as an initial upper bound. As the computations proceed, feasible solutions (those satisfying the capacity constraints) are discovered from time to time, and the best one yet found is retained as an incumbent solution with the corresponding value as the best upper bound. It may happen that for a given partial solution $S_\ell$ we can determine a best completion of $S_\ell$ , i.e., a feasible completion that minimizes the objective function value among all feasible completions of $S_\ell$ . If such a best feasible completion is better than the best upper bound, then it replaces the latter. Or we may be able to determine that $S_\ell$ has no feasible completion better than the incumbent. In either case, we can fathom $S_\ell$ . (Various situations of fathoming and back-

tracking in our branch-and-bound procedure are described in the following discussion.) All completions of a fathomed partial solution $S_\ell$ have been implicitly enumerated in the sense that they can be excluded from further consideration (with the exception of the relevant best feasible solution of $S_\ell$ if it has been retained as the best upper bound).

In our branch-and-bound procedure, at any given node where we can fathom $S_\ell$, we backtrack to the parent node and move to the right-hand branch (if that branch has not already been explored) by fixing the appropriate $x_{ij}$ variable at $0$. However, if the right-hand branch has already been explored, we continue backtracking to a parent node where we can move to a right-hand branch. For example, in Figure 4a, when backtracking from node 3, we move to the parent node 2, and to the right to node 4 by setting $x_{42} = 0$. However, when backtracking from node 6, we move back to node 4, then back to node 2, then back to node 1, and to the right to node 7 by setting $x_{13} = 0$.

On the other hand, if the partial solution $S_\ell$ cannot be fathomed, we branch to the left and augment $S_\ell$ by fixing a free variable $x_{ij}$ at 1 (based on the branching rule), and then we try to fathom the resulting partial solution. In addition to the one variable selected by the branching rule, some other free $x_{ij}$ variables can also be fixed at 0 or 1 according to the application of rules other than the branching rule. Note that this can also happen when backtracking, i.e., when $S_\ell$ has been fathomed and we backtrack and move to the right by setting the appropriate $x_{ij}$ variable to $0$.

Let us consider examples of both situations, i.e., when $S_\ell$ has not been fathomed and when $S_\ell$ has been fathomed. In Figure 4a we cannot fathom $S_1$ (i.e., $S$ at node 1), so we move to node 2 by

augmenting $S_1$ by fixing $x_{13} = 1$ based on the branching rule, and by fixing $x_{31} = 0$ and $x_{41} = 0$ based on the application of the other rules. Similarly, we move from node 2 to node 3 by augmenting $S_2$ by fixing $x_{42} = 1$ . As an example of backtracking, when we fathom $S_3$ , we move back to the parent node 2, and to the right to node 4, getting a new partial solution $S_4$ by replacing $x_{42} = 1$ with $x_{42} = 0$ , and further augmenting it by fixing $x_{21} = 0$ and $x_{11} = 1$ based on the application of the other rules.

Computationally, the storage and update of partial solution $S_\ell$ is easily accomplished by considering Figure 4b. If, at a given node, the partial solution $S_\ell$ has not been fathomed, e.g., at node 4, determine the next branching variable by using the branching rule, i.e., $x_{24}$ , and augment $S_4$ by adding 204 as the last entry. Also, augment $S_4$ with any other free $x_{ij}$ variables, if appropriate, depending on the application of the other rules. Now, consider the case where the partial solution $S_\ell$ has been fathomed, e.g., at node 6, and we backtrack; starting with the last entry in $S_\ell$ . we consider one entry at a time, going backwards, until we find a positive number which is not underlined. In our example, it is 103. In other words, we must branch to the right by fixing $X_{13} = 0$ , i.e., we replace 103 with - 103 and we are at node 7. Should we find that we have no positive number, the procedure terminates since we are back at the root node and the right branch has already been explored. This happens when backtracking from node 7.

In the branch-and-bound procedure we generate a sequence of partial solutions as we move from one node to another. This sequence is non-redundant in the sense that no completion of a partial solution ever duplicates a completion of a previous partial solution that has been fathomed.

Since one of the $x_{ij}$ values, for each $j$, must be 1, a total of $(2m-1)^n$ nodes are theoretically possible for complete enumeration. However, most of the solutions may be infeasible because of the capacity constraints. The branch-and-bound procedure, through a judicious choice of branching variables, and elimination of certain infeasible and non-optimal assignments through various rules, turns out to be a practical and computationally efficent algorithm. The various components of this procedure are described next. Detailed procedural steps and the solution of a test problem will be covered in Chapter 4.

## 3.1 Bounds

### 3.1.1 Lower Bound

At a given node $\ell$ in the branch-and-bound tree, a lower bound (LOWB) is obtained by solving relaxed problem $(PR_\ell)$.

$$LOWB = Z(PR_\ell) \tag{40}$$

Recall that problem $(PR_\ell)$ is very easy to solve by considering the minimum $c_{ij\ell}$ over those j's for which $x_{ij}$ is not fixed at 1, i.e., $j\varepsilon\bar{W}$, where $\bar{W}$ is the complement of $W$ defined by expression (32).

$$Z(PR_\ell) = \sum_{j\varepsilon W} c_{ij\ell} + \sum_{j\varepsilon\bar{W}} \min_{\substack{i \\ (i,j)\varepsilon\bar{S}}} c_{ij\ell} + FC_\ell \, , \tag{41}$$

where $c_{ij\ell}$ is given by expression (38), i.e.,

$$c_{ij\ell} = a_{ij} + \sum_k b_k \, (1 - \delta_{k\ell}) \, r_{ijk} \, , \tag{38}$$

and the fixed cost $(FC_\ell)$ is given by expression (39), i.e.,

$$FC_\ell = \sum_k \delta_{k\ell} \, b_k \, , \tag{39}$$

where $\delta_{k\ell}$ is given by expression (34).

Note that if none of the $x_{ij}$ variables is fixed at 1 , as is generally the case at the root node, then all $\delta_{kl} = 0$ , and, therefore, $FC_1 = 0$ , and $c_{ij1} = a_{ij} + \sum_{k=1}^{p} b_k r_{ijk}$ . $Z(PR_1)$ is, then, simply the middle part of expression (41). We use the term "generally" because it is possible that the capacity rule could force certain $x_{ij}$ variables to 1 (or 0) at the root node, prior to solving the relaxed problem $(PR_1)$ .

### 3.1.2 Upper Bound

At any given node $\ell$ , let $X = \{x_{ij}\}$ represent the solution of problem $(PR_\ell)$ . If this solution is feasible for problem (P), i.e., if $X$ satisfies the capacity constraints (5) or (5')

$$\sum_{\substack{i \\ x_{ij} \in X}} \sum_{j} d_{ijk} x_{ij} \leq s_k y_k \qquad \forall k , \tag{42}$$

$$\text{where } y_k = 1 \text{ if } \sum_{\substack{i \\ x_{ij} \in X}} \sum_{j} d_{ijk} x_{ij} > 0 , \qquad = 0 \text{ otherwise ,} \tag{43}$$

then the value of problem (P) corresponding to this solution gives an upper bound (UPB):

$$UPB = \sum_{\substack{i \\ x_{ij} \in X}} \sum_{j} a_{ij} x_{ij} + \sum_{k} b_k y_k , \tag{44}$$

where $y_k$ is defined by (43) .

### 3.1.3 Best Upper Bound

A current lowest upper bound is retained as the best upper bound (BUB), the corresponding solution X representing the incumbent solution.

The branch-and-bound procedure is initiated by assuming a very large value as the best upper bound, and is replaced by better (lower) values as the procedure continues.

A positive fractional value $\varepsilon$ can be specified if a sub-optimal solution is acceptable. For example, for $\varepsilon = 0.001$, the resulting solution value is guaranteed to be within 0.1 percent of the optimal solution value. When $\varepsilon$ is non-zero, the adjusted best upper bound (BUBS) is defined as:

$$\text{BUBS} = \text{BUB}/(1 + \varepsilon) \ . \tag{45}$$

Obviously when $\varepsilon = 0$, BUBS = BUB .

## 3.2 Facility Usage Rule

This rule is used to identify facilities forced into usage at a given node $\ell$ and hence fix corresponding free variables $y_k$ at 1 .

For a partial solution $S_\ell$, define

$$\begin{aligned} \bar{d}_{jk\ell} &= d_{ijk} \quad \text{if} \quad j \varepsilon W , \\ &= \min_{\substack{i \\ (i,j)\varepsilon \bar{S}}} d_{ijk} \quad \text{if} \quad j \varepsilon \bar{W} . \end{aligned} \tag{46}$$

The facility usage rule states that for any facility $k$, where $y_k$ is not already fixed at 1, if $\sum_j \bar{d}_{jk\ell} > 0$, then facility $k$ is forced into usage and, therefore, $y_k$ should be fixed at 1 .

This rule is applied at every node prior to applying the capacity rule. In other words, this rule is applicable to capacitated as well as uncapacitated problems.

## 3.3 Capacity Rule

This rule is designed to "exclude" infeasible assignments prior to solving the relaxed problem $(PR_\ell)$. This is done by exploiting the

relationship between the capacities required $(d_{ijk})$ and the capacities available $(s_k)$ for a given partial solution of problem (P).

The capacity rule states that for a facility $k$ and an activity $j$, "exclude" a free $x_{ij}$ variable (i.e., fix it at 0) for which

$$(d_{ijk} - \bar{d}_{jk\ell}) > (s_k - \sum_j \bar{d}_{jk\ell}) \ , \qquad (i,j)\in\bar{S} \tag{47}$$

where $\bar{d}_{jk\ell}$ is defined by expression (46). The right-hand side of this inequality (47), when positive, represents the available capacity at facility $k$. The left-hand side shows, for a given $j$, the difference between a $d_{ijk}$ corresponding to a free $x_{ij}$ variable and $\bar{d}_{jk\ell}$. If, for a specific $d_{ijk}$, this difference is more than the available capacity, the corresponding free $x_{ij}$ variable, if fixed at 1, would result in an infeasible solution. Thus, by looking ahead, we can exclude such a free $x_{ij}$ variable by assigning it a value of 0.

Note that if the right-hand side of expression (47) is negative, then any completion of such a partial solution will be infeasible and we backtrack in our branch-and-bound procedure.

The capacity rule is applied to all the facilities by considering one facility at a time. The cycle of examining all the facilities continues until no more assignments can be excluded. During the course of application of this rule, if all but one of the free $x_{ij}$ variables have been excluded (fixed at 0) for a given $j$, then that particular $x_{ij}$ variable is fixed at 1 because of constraints (2), i.e., each activity $j$ must be assigned to one and only one design $i$. The partial solution is updated accordingly to reflect the $x_{ij}$ variables fixed at 0 or 1 due to the application of the capacity rule.

The capacity constraints for an uncapacitated problem are not active. Hence, the capacity rule is useful only for capacitated problems.

### 3.4 Branching Rule

This rule provides the choice of the $x_{ij}$ variables on which to branch. If the partial solution at a given node $\ell$ is not fathomed, we branch further by fixing a free $x_{ij}$ variable at 1 and moving to the left branch node.

According to the branching rule the choice of the branching variable depends on the $c_{ij\ell}$ values and is such that the corresponding $x_{ij}$, if perturbed, has the maximum impact on the optimal value of problem $(PR_\ell)$.

For a given $j$, define $c_{i_1j\ell}$, the minimum permissible $c_{ij\ell}$, and $c_{i_2j\ell}$, the second smallest permissible $c_{ij\ell}$, i.e.,

$$c_{i_1j\ell} = \min_i c_{ij\ell} \quad \text{for} \quad j\varepsilon\bar{W} \quad \text{and} \quad (i,j)\varepsilon\bar{S} \tag{48}$$

$$\text{and} \quad c_{i_2j\ell} = \min_{\substack{i \\ i \neq i_1}} c_{ij\ell} \quad \text{for} \quad j\varepsilon\bar{W} \quad \text{and} \quad (i,j)\varepsilon\bar{S} \tag{49}$$

$$\text{For each} \quad j\varepsilon\bar{W}, \text{ define } D_{j\ell} = c_{i_2j\ell} - c_{i_1j\ell}. \tag{50}$$

Our branching rule states that a free $x_{ij}$ variable corresponding to $c_{i_1j\ell}$ such that $D_{j\ell}$ is maximized over all $j$, is selected as the next branching variable and assigned a value of 1.

### 3.5 Bounding Rule

This rule is designed to "exclude" certain non-optimal assignments. These assignments cannot lead to an optimal solution as we branch from one node to the next left branch node.

The bounding rule states that a free $x_{ij}$ variable should be excluded (by assigning it the value 0) for which

$$(c_{ij\ell} - c_{i_L j\ell}) > (\text{BUBS} - \text{LOWB}) \quad \text{for} \quad j\varepsilon\bar{W} \quad \text{and} \quad (i,j)\varepsilon\bar{S} \tag{51}$$

where $c_{i_L j\ell}$ , BUBS, and LOWB are given by expressions (48), (45), and (40), respectively.

Thus, by looking ahead, we exclude those assignments which will provide lower bounds higher than BUBS.

The bounding rule is applied to each $j\varepsilon\bar{W}$ just prior to selecting the $x_{ij}$ variable for branching to the left.

As in the case of the capacity rule, if the bounding rule results in excluding (fixing at 0) all but one of the free $x_{ij}$ variables for a given $j\varepsilon\bar{W}$ , then that particular $x_{ij}$ variable is fixed at 1 . Also the partial solution is updated accordingly to reflect the $x_{ij}$ variables fixed at 0 or 1 due to the application of the bounding rule.

### 3.6 Backtracking Rules

If a partial solution at a given node has been fathomed, we backtrack. The backtracking rules are typical of a branch-and-bound procedure. In addition, the application of the capacity rule and the bounding rule can lead to backtracking. The criteria for backtracking include the following.

(a) When applying the capacity rule, if the available capacity given by the right-hand side of inequality (47) is negative, i.e., $(s_k - \sum_j \bar{d}_{jk\ell}) < 0$ , then backtrack.

(b) If LOWB $\geq$ BUBS, then backtrack. Otherwise compute UPB if the solution is feasible in problem (P). Then update BUB and BUBS if UPB < BUB; and backtrack if LOWB = BUBS.

(c) If further branching is not possible, then backtrack. This can happen due to the capacity rule, the bounding rule, or the branching rule if the updated partial solution is such that no further branching is possible, i.e., $x_{ij}$ variables are fixed at 1 for all $j$ , or equivalently, $\bar{W} = \phi$ .

When any of the backtracking criteria apply, we backtrack to the parent node and move to the right branch node (if the right branch has not already been explored) by fixing the appropriate $x_{ij}$ variable at 0 . If the right branch has already been explored, we continue backtracking to a parent node where we can move to a right branch node. The branch-and-bound procedure terminates when we backtrack to the root node and find that the right branch node has already been explored.

## 4. COMPUTATIONAL STEPS AND THE COMPUTER PROGRAM

A computer program called ZIPCAP (an acronym for Zero-one Integer Program for multiactivity multifacility Capacity-constrained Assignment Problems) implementing the branch-and-bound methodology has been developed.

Detailed procedural steps and guidelines to use the computer program are described in a separate document [Chhabra and Soland (1980)] titled "Program Description and User's Guide for ZIPCAP--a Zero-one Integer Program to solve multiactivity multifacility Capacity-constrained Assignment Problems." Specifically, the document includes:

- Problem formulation (P) and potential areas of application

- Overall flow diagram and detailed procedural steps for the computer program

- Program listing and dictionary of the symbolic names. The listing includes extensive use of comment cards to explain various computational steps.

- User information including

  - schematic diagram of the deck structure,

  - detailed instructions for the job control (JCL) cards, program parameter card, program options card, and the various other input data cards.

- Three test problems to demonstrate the use of the program. The display includes coded input and annotated outputs reflecting the use of selected program options.

As mentioned earlier, ZIPCAP is primarily designed for capacitated problems. However, uncapacitated problems can be solved as a special case, and this is demonstrated by including an uncapacitated test problem.

Because of the extensive coverage of the program description and user guidelines in the above document, this chapter provides only an overview of the computer program, including an overall flow diagram, and a summary of the program options, in order to provide continuity in this document. In addition, a step-by-step description of a test problem is presented to demonstrate the use of the various components of the branch-and-bound methodology. The computer printout showing step-by-step details is obtained by use of one of the program options. The use of this option to display detailed steps in this document, in fact, complements the use of the various options demonstrated in the other document.

## 4.1 The Program

Figure 5 presents a simplified flow diagram of the branch-and-bound procedure. The major computational steps for the computer program are numbered in circles. These steps are essentially based on the methodology components described in the previous chapter. A step-by-step description has been included in the other document [Chhabra and Soland (1980)].

The computer program ZIPCAP is written in FORTRAN IV, and has been developed and tested on an IBM 3031 at the George Washington University. The program, comprising about 480 lines is currently dimensioned for a maximum problem size of 35 designs (m), 35 activities (n) and 30 facilities (p). The program size to execute a problem has two components: one, due to the program itself, comprising 173 K bytes, and the other dependent on the dimensions of the arrays given by the following functional relationship.

$$f(m,n,p) = 4[(p+4)mn + (m+5)p+9n] \text{ bytes}$$

The computer program listed in the other document has since been further improved. The basic improvement has been the addition of the facility usage rule. This rule, as described in Chapter 3, is applied both to capacitated and uncapacitated problems just before the application of the capacity rule. For completeness of this document, a revised program listing is included in Appendix A. It may be mentioned that

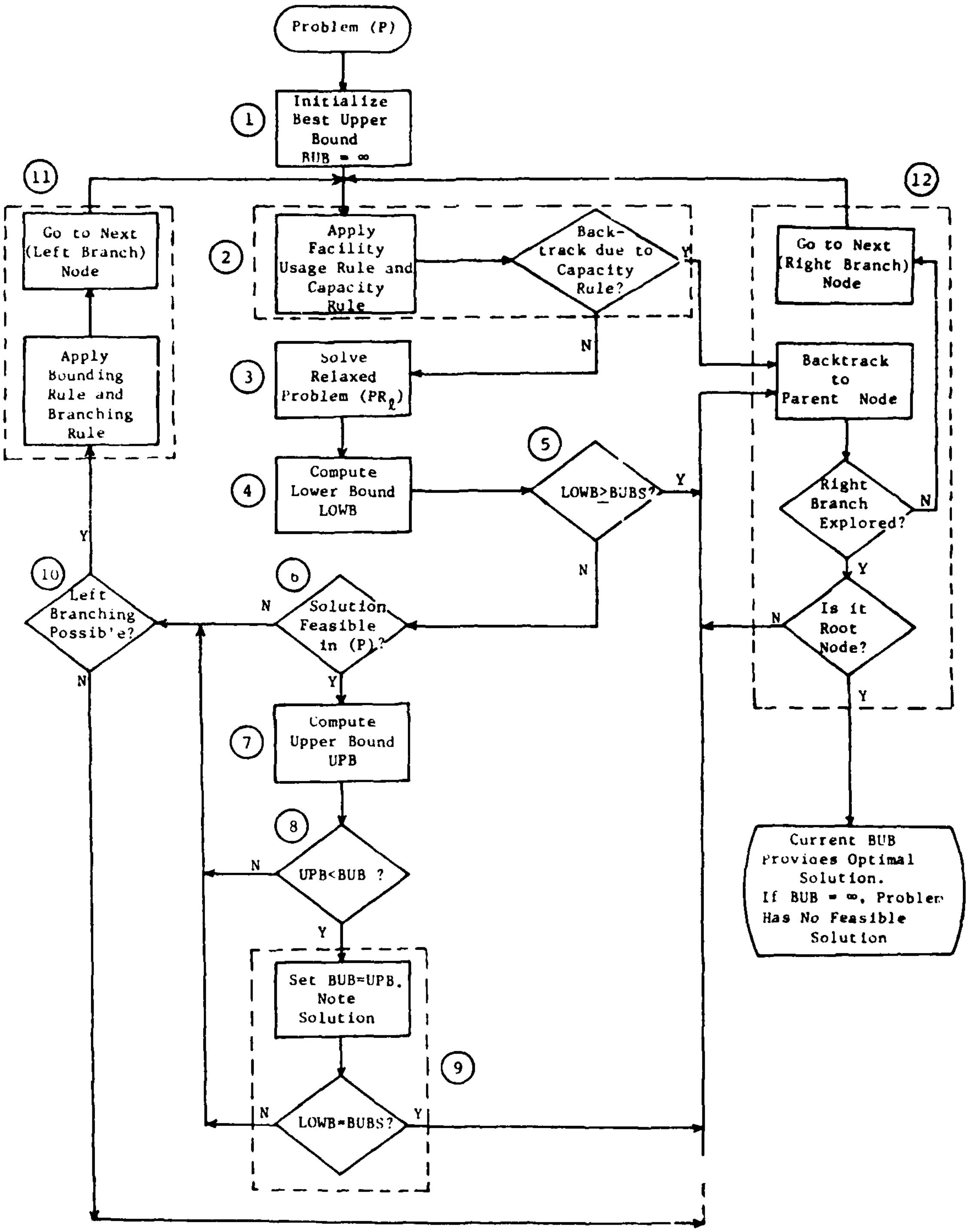



Figure 5.

Simplified flow diagram for the branch-and-bound procedure

the revised program solves the test problems included in the other document more efficiently -- in less time and in fewer nodes (with an average reduction in nodes of 31 percent). The improvement in efficiency seems to result from the "multiplicative" effect of the various rules. Another improvement made is that the computer printout always displays the node number (IBNOD) at which the best upper bound changes (improves) and the corresponding values of the best upper bound (BUB) and the adjusted best upper bound (BLBS).

ZIPCAP provides numerous options to the program user. These options, described in the other document, are summarized in Table 3.

Option ICAPR, the capacity rule, is automatically skipped by the program when solving an uncapacitated problem. Option ISTEP, the intermediate steps' listing, even when skipped, provides information on the total number of nodes explored. A summary listing provides necessary information to construct the branch-and-bound tree, whereas a detailed listing of the intermediate steps is useful when changing or debugging the program.

Option EPS, the optimal/suboptimal solution, provides the flexibility of obtaining a suboptimal value guaranteed to be within a specified fraction of the optimal value. The resulting solution may be suboptimal but could provide a considerable saving in terms of exploring fewer nodes in comparison to those necessary for obtaining an optimal solution.

Option ET, by providing important information at a specified elapsed time, is useful in a situation where the total time allocated to solve a problem may not be sufficient and the program terminates before verifying an optimal solution. The information provided by this option includes an updated partial solution showing the $x_{ij}$ variables fixed at 0 or 1, at the current node being explored at the specified tire ET. By looking at the first few variables displayed in the partial solution of the current node, it is possible to assess the extent of the branch-and-bound tree explored until time ET. For

Table 3

SUMMARY OF ZIPCAP OPTIONS

| Option Name | | | Alternatives Available to User |
|---|---|---|---|
| 1. | IINPT | -- Input Listing | . List input data<br>. Skip this option |
| 2. | ICAPR | -- Capacity Rule | . Use capacity rule<br>. Skip this option |
| 3. | ISTEP | -- Intermediate Steps Listing | . Skip listing of intermediate steps<br>. Provide a summary of intermediate steps<br>. Provide detailed intermediate steps |
| 4. | IUNCAP | -- Capacitated/Uncapacitated Problem | . Solve a capacitated problem<br>. Solve an uncapacitated problem |
| 5. | EPS | -- Optimal/Suboptimal Solution | . Optimal solution<br>. Suboptimal solution acceptable within a specified fractional value (epsilon) |
| 6. | ET | -- Information at a specified Elapsed Time | . Provide the following information at elapsed time ET:<br>- Best upper bound, corresponding solution, and the node at which found<br>- node being explored and detailed steps for that node<br>. Skip this option |

example, in view of the terminology in Figure 4b (Chapter 3), if, at an arbitrary node, the first term of the partial solution is positive, i.e., the $x_{ij}$ variable has value 1, then we are still in the left half of the total branch-and-bound tree. If the first term is negative, i.e., the $x_{ij}$ variable has value 0, then we are in the right half of the total branch-and-bound tree and have explored half of the total (theoretical) solutions corresponding to the left half of the tree. If the first two terms are negative, i.e., the first two $x_{ij}$ variables have value 0, then one quarter of the total (theoretical) solutions remain to be explored, since we are in the next right half of the right half of the total branch-and-bound tree, as illustrated in Figure 6.

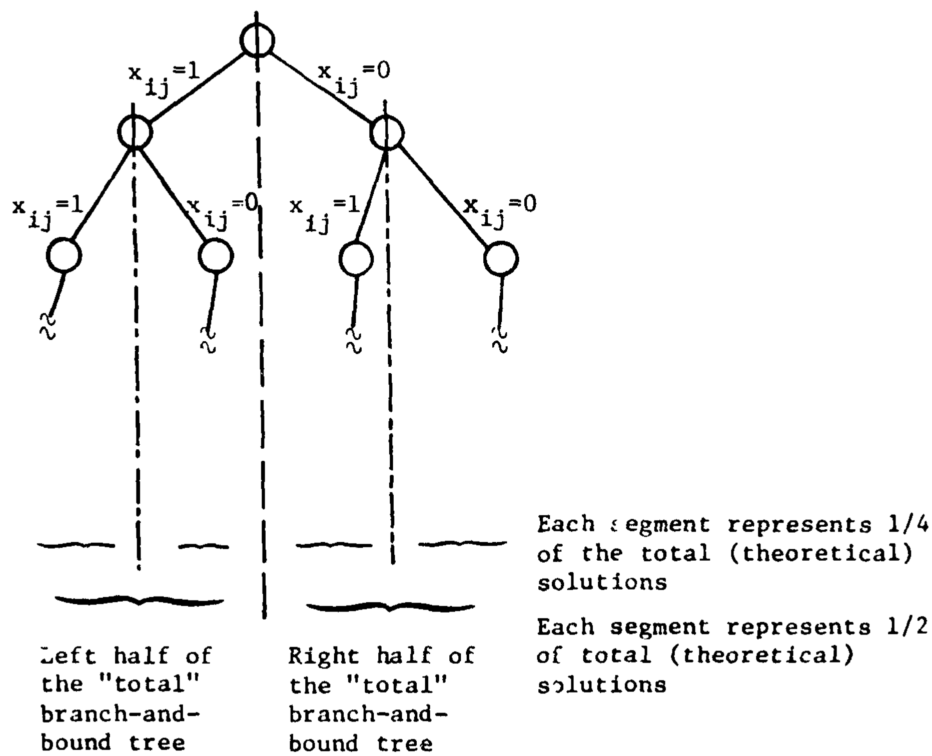



Figure 6

Illustration for estimating the extent of the branch-and-bound tree explored

Recall from Chapter 3, that a total of $(2m-1)^n$ nodes are theoretically possible. Thus, if the first $g$ $[g \leq (m-1)n]$ terms at an arbitrary node are negative, then theoretically about $[(2m-1)^n/_2g]$ nodes remain to be explored.

## 4.2 An Illustrative Example

We consider a capacitated test problem with five designs (m), four activities (n), and eight facilities (p) to demonstrate the use of the branch-and-bound procedure and the computer program.

The computer printout for this problem showing step-by-step details for a couple of nodes is presented in Appendix B.

As shown in the beginning of the printout, the options selected are:

- IINPT = 1, i.e., list the input data
- ICAPR = 1, i.e., use the capacity rule
- ISTEP = 2, i.e., list detailed intermediate steps
- IUNCAP = 1, since this is a capacitated problem
- EPS = 0.0 implying that an optimal solution is desired
- ET = 0.0 since a detailed listing of intermediate steps will be available.

Following the listing of the options, input data listed for the problem include variable costs $a_{ij}$, fixed costs $b_k$, available capacities $s_k$, and capacities required $d_{ijk}$. The $e_{ik}$ values are generated by the computer program.

The computer program follows the procedural steps marked in the flow diagram presented in Figure 5. These steps, along with the relevant terminology used in the computer printout, are described below for a couple of nodes, followed by a complete branch-and-bound tree for this problem. As mentioned earlier, a dictionary of the symbolic names used in the computer program is included in the other document.

Node 1

Step 1: Initialize.

Initialize BUB = 9999999.0, and since EPS = 0.0, BUBS = BUB. Also $S = \phi$ and $W = \phi$ . In the computer printout, vector FIX(J) represents the set W, and matrix CX(I,J) represents both, fixed and free $x_{ij}$ variables. In the CX(I,J) matrix, an $x_{ij}$ variable fixed at 1 or 0 is represented as 1 or 2, respectively, and a free $x_{ij}$ variable is represented by the value 0. Initially, all the $x_{ij}$ variables are free as shown by matrix CX(I,J) in the printout.

Step 2: Apply the facility usage rule and the capacity rule for $k=1,2,\ldots,8$ .

In the printout, MIND(J) represents $\bar{d}_{jk\ell}$ defined by expression (46), and MINSD represents $\sum_j \bar{d}_{jk\ell}$ . As shown in the printout, MINSD is 0 for $k=1,2,\ldots,8$ , and so the facility usage rule does not force any facilities into usage; and as shown by matrix CX(I,J) for $k=1,2,\ldots 8$ , the capacity rule does not fix any $x_{ij}$ variables.

Step 3: Solve the relaxed problem $(PR_1)$ .

In the printout FLB(K) represents $\delta_{k\ell}$ , given by expression (34), for computing $FC_\ell$ , and C(I,J) represents $c_{ij\ell}$ defined by expression (38). Being at the root node, $\ell = 1$. Further the solution of problem $(PR_\ell)$ , i.e., $X = \{x_{ij}\}$ is shown in the printout by SOLX(J) which for $(PR_1)$ is $X = \{x_{41} = x_{42} = x_{23} = x_{44} = 1\}$ .

Step 4: Compute the lower bound.

The expressions (40) and (41), i.e.,

$$LOWB = Z(PR_\ell) \tag{40}$$

$$= \sum_{j\in W} c_{ij\ell} + \sum_{j\in \bar{W}} \min_{\substack{i \\ (i,j)\in \bar{S}}} c_{ij\ell} + FC_\ell \tag{41}$$

are represented in the printout as

$$LOWB = MINSC + FC$$
$$= 729839.3125 + 0 = 729839.3125$$

Step 5: Compare LOWB with BUBS.

Since LOWB < BUBS, go to Step 6

Step 6: Check if solution X is feasible in problem (P), i.e., expression (42) is satisfied.

$$\sum_{\substack{i \\ x_{ij} \varepsilon X}} \sum_{j} d_{ijk}\, x_{ij} \leq s_k y_k \qquad \forall k \tag{42}$$

In the printout, NSUMD represents the left-hand side of this inequality, and for each $k$, the capacity constraints are satisfied.

Step 7: Compute the upper bound.

UPB is given by expression (44), i.e.,

$$UPB = \sum_{\substack{i \\ x_{ij} \varepsilon X}} \sum_{j} a_{ij}\, x_{ij} + \sum_{k} b_k y_k \tag{44}$$

In the printout, the corresponding expression is represented as

$$UPB = NSUMA + FCUB$$
$$= 678{,}502 + 101{,}000 = 779502.0 \; .$$

Step 8: Compare UPB with BUB.

Since UPB < BUB, go to Step 9.

Step 9: Set BUB = 779502.0 . Since EPS = 0.0, BUBS = BUB.

Since LOWB < BUBS, go to Step 10.

Step 10: Left branching is possible since $W = \phi$ as shown by vector FIX(J); go to Step 11.

Step 11: Apply the bounding rule and the branching rule.

According to our bounding rule, a free $x_{ij}$ variable is excluded

(fixed at 0) for which

$$(c_{ij\ell} - c_{i_1 j\ell}) > (\text{BUBS} - \text{LOWB}) \text{ for } j\varepsilon\bar{W} \text{ and } (i,j)\varepsilon\bar{S} \qquad (51)$$

For $x_{13}$ , (210,381.4375-145,201.5) > (779,502.0-729,839.3125). This also holds for $x_{33}$ and $x_{14}$ , i.e., the bounding rule results in fixing $x_{13}$ , $x_{33}$ , and $x_{14}$ at 0. This is shown in the printout by matrix CX(I,J) where the corresponding variables have been assigned the value 2 because of the bounding rule.

The branching rule directs us to select a free $x_{ij}$ variable corresponding to $c_{i_1 j\ell}$ for which $D_{j\ell} = c_{i_2 j\ell} - c_{i_1 j\ell}$ is maximized over all j . In the printout, $c_{i_2 j\ell}$ , $c_{i_1 j\ell}$ , and $D_{j\ell}$ are represented by NMINC(J), MINC(J) and DIFBR(J), respectively. Since $D_{21}$ is the maximum, $x_{42}$ is selected as the next left branching variable. This is shown in the printout by BR1 and is represented as (100 i + j) e.g., 402 .

Using the terminology employed in Figures 4a and 4b, the $x_{ij}$ variables fixed at 0 or 1 in the partial solution $S_1$ will be shown as $S_1$ = {- 103, - 303, - 104, 402} . In the computer printout, vector STX displays the $x_{ij}$ variables fixed at 0 or 1. The representation of the variables is, however, somewhat different. An $x_{ij}$ variable fixed at 0, due to any rule, is shown as - (100 i + j) - 1,000,000, e.g., $x_{13}$ is shown as - 1,000,103; an $x_{ij}$ variable fixed at 1 due to the branching rule is represented as (100 i + j), e.g., $x_{42}$ as 402; and an $x_{ij}$ variable fixed at 1 due to a rule other than the branching rule is shown as (100 i + j) + 1,000,000, e.g., $x_{23}$ is represented as 1,000,203.

In the printout, vector STX represents updated partial solution $S_1$.

We now move to Node 2.

Node 2

The updated matrix CX(I,J) and vector FIX(J) are displayed in the printout.

Step 2: Apply the facility usage rule and the capacity rule for $k=1,2,\ldots,8$ .

As shown in the printout, MINSD (representing $\sum_j \bar{d}_{jk\ell}$) , being positive for $\ell=1,2,3,4$, and 5 , these facilities are forced into usage. Further, for $k=4$ , expression (47) holds for $x_{34}$ and $x_{54}$ , i.e.,

$$(180-0) > (200-30), \text{ and}$$

$$(180-0) > (200-30), \text{ respectively.}$$

As shown by matrix CX(I,J) in the printout, these two variables are excluded (fixed at 0) by the capacity rule. Since the capacity rule results in fixing at least one variable in the first cycle, another cycle is repeated as displayed in the printout. The second cycle does not fix any more variables. Vector STX is updated accordingly.

Step 3: Solve the relaxed problem $(PR_2)$ .

$\delta_{k2}$ represented by FLB(K) , $c_{ij2}$ represented by matrix C(I,J), and solution $X$ represented by SOLX(J) are displayed in the printout.

Step 4: Compute LOWB.

LOWB, from the printout, is equal to 749011 4375.

Step 5: Compare LOWB with BUBS.

Since LOWB < BUBS, go to Step 6.

Step 6: Check if solution X is feasible in (P).

In the printout, for $k=4$ , NSUMD = 290 > 200 , i.e., expression (42) is not satisfied, and we go to Step 10.

Step 10: As shown by vector FIX(J), left branching is possible and we go to Step 11.

Step 11: Apply the bounding rule and the branching rule.

As displayed by matrix CX(I,J) in the printout, the bounding rule results in fixing $x_{21}$ and $x_{24}$ at 0. Now, for $j=4$, except for $x_{44}$, all the $x_{ij}$ variables are fixed at 0; therefore $x_{44}$, is fixed at 1. This is reflected by matrix CX(I.J), and vector FIX(J). Vector STX is updated accordingly.

The branching rule selects $x_{41}$ as the next branching variable. This is shown in the printout by BR1, and vector STX is updated accordingly.

We now move to Node 3.

Node 3:

The updated matrix CX(I,J) and vector FIX(J) are displayed in the printout.

Step 2: Apply the facility usage rule and the capacity rule for $k=1,2,...,8$.

The facility usage rule forces facilities 1 to 5, and 8 into usage. For $k=4$, the capacity rule excludes $x_{43}$ and $x_{53}$, i.e., fixes them at 0; and for $j=3$, all but $x_{23}$ being fixed at 0, $x_{23}$ is fixed at 1. This is displayed in the printout by matrix CX(I,J) and vector FIX(J). Vector STX is updated accordingly.

Although the capacity rule has fixed at least one $x_{ij}$ variable during the initial cycle, another cycle is not necessary, as displayed by vector FIX(J) which represents set W, since we have an $x_{ij}$ variable fixed at 1 for each of the $n$ columns (activities).

Step 3: Solve the relaxed problem $(PR_3)$.

SOLX(J) displays the solution for the relaxed problem.

Step 4: Compute LOWB.

LOWB, shown in the printout, is equal to 779502.0 .

Step 5: Compare LOWB with BUBS.

Since LOWB = BUBS, go to Step 12.

Step 12: Backtrack.

We backtrack by moving to the parent Node 2, and branching to the right by setting $x_{41} = 0$ (since the right branch has not yet been explored). In the printout, this is accomplished by observing the last entry in vector STX, and moving backwards, one entry at a time, until we find a positive entry without 1,000,000 added to it. The corresponding $x_{ij}$ variable is fixed at 0, and we move to the right branch node. Matrix CX(I,J), vector FIX(J) and vector STX are updated accordingly. As displayed in the printout, entry 401 in vector STX is such an entry, and variable $x_{41}$ is fixed at 0 for branching to the right. This is shown in the printout by BRO as 401. The updated vector STX is also displayed.

We now move to Node 4.

<u>Node 4</u>

The updated matrix CX(I,J) and vector FIX(J) are displayed in the printout.

Step 2: Apply the facility usage rule and the capacity rule for $k=1,2,\ldots,8$ .

As displayed in the printout, for k=4, MINSD=230 > 200 , i.e., the right-hand side of inequality (47), $(s_k - \sum_j \bar{d}_{jk\ell}) < 0$ , and according to our backtracking rules, we backtrack, i.e., go to Step 12.

Step 12: Backtrack.

We backtrack to the parent Node 2, and since the right-hand branch has already been explored, backtrack to Node 1 and to the right-hand branch by fixing $x_{42}$ to 0 . This is shown in the printout by BRO as 402, and vector STX is updated accordingly.

We now move to the next node, i.e., Node 5.

Branch-and-Bound Tree

We continue the branch-and-bound procedure from one node to another until we backtrack to the root node and find that the right branch has already been explored. The procedure, then, terminates and the solution corresponding to the best upper bound is the optimal solution.

For this problem, a total of nine nodes are explored and the optimal value equals 779502.0. The optimal solution is $x_{41} = x_{42} = x_{23} = x_{44} = 1$ and $y_1 = y_2 = y_3 = y_4 = y_5 = y_8 = 1$. This is displayed in the computer printout on the last page of Appendix B.

Figure 7a presents the branch-and-bound tree for this problem, and shows the node numbers, the bounds, and the branching variables.

In order to demonstrate the role of the capacity rule and the bounding rule, Figure 7b displays the $x_{ij}$ variables fixed as 0 or 1 by these rules for this test problem.

The cumulative effect of the various rules, including the facility usage rule, the capacity rule, and the bounding rule, makes the branch-and-bound procedure quite efficient. Further, the storage and updating of the $x_{ij}$ variables fixed at 0 or 1 is done in a manner that makes utmost use of the relevant information at the preceding node.

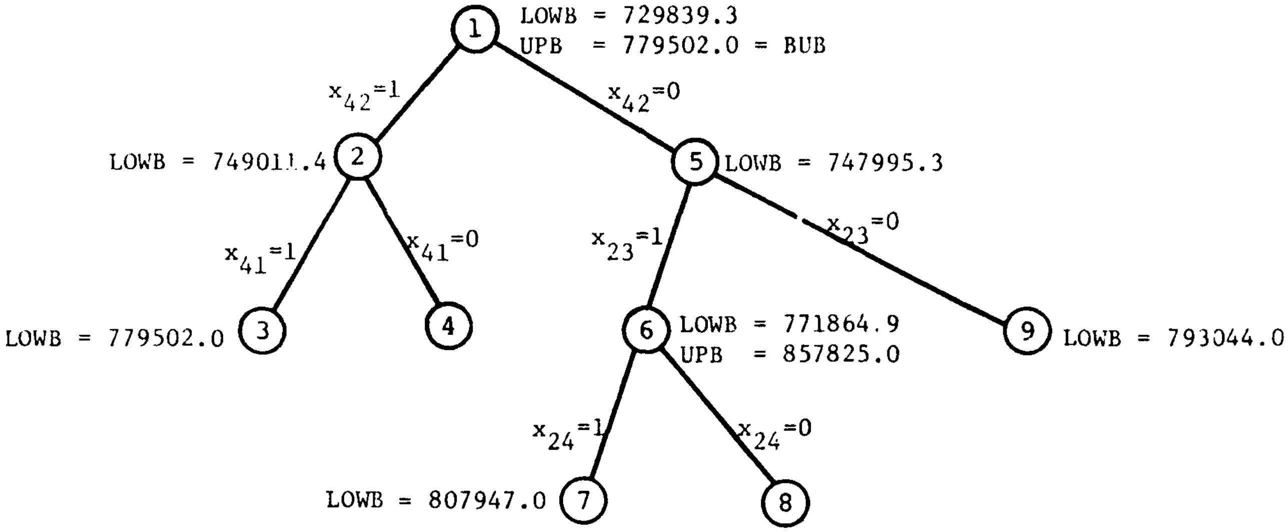



Figure 7a

Branch-and-bound tree for a test problem
(Test Problem with m=5, n=4, and p=8)

| Node | Capacity Rule | Bounding Rule |
|---|---|---|
| 1 | | $x_{13}=0$ , $x_{33}=0$ , $x_{14}=0$ |
| 2 | $x_{34}=0$ , $x_{54}=0$ | $x_{21}=0$ , $x_{24}=0$ , $x_{44}=1$ |
| 3 | $x_{43}=0$ , $x_{53}=0$ , $x_{23}=1$ | |
| 5 | | $x_{43}=0$ , $x_{34}=0$ |
| 6 | | $x_{11}=0$ , $x_{21}=0$ , $x_{31}=0$ , $x_{22}=0$ , $x_{54}=0$ |
| 8 | $x_{44}=1$ | |
| 9 | $x_{53}=1$, $x_{44}=0$, $x_{54}=0$, $x_{24}=1$ | |

Figure 7b

Variables fixed by the capacity rule and the bounding rule

## 5. COMPUTATIONAL TEST RESULTS

The computer program ZIPCAP has been tested on several problems. Although primarily designed for capacitated problems (i.e., where the capacity constraints are active), the program can also be used for solving uncapacitated problems as a special case. Since the data available for capacitated problems were limited, some uncapacitated problems were also considered for testing the program. (Most of the data were furnished by Professor Pinkus and are related to his work on multi-echelon inventory systems.)

Table 4 presents the test results of ZIPCAP. In order to verify the optimal solutions, the test problems were also solved by using the 0-1 integer programming code RIP30C [Geoffrion and Nelson (1968)].

In the table, the problem size shows the number of designs (m), activities (n), and facilities (p). This is equivalent to solving a problem having $mn+p$ variables and $n+p$ constraints. The elapsed time represents the time in seconds to solve the problem, excluding the time to read and write the input and to write the output. The total number of nodes explored by ZIPCAP for a specified set of options is also shown.

Both RIP30C and ZIPCAP were run on an IBM 3031 at The George Washington University. The last problem in the table was not run using RIP30C because of the code's capacity limitation to 90 variables and 50 constraints.

The test problem with $m=3$ , $n=4$ , and $p=5$ has three variations, using different values for the facility capacities. The data for the variable costs $a_{ij}$ , fixed cost $b_k$ , and the capacity requirements $d_{ijk}$ are given in the other document, i.e., Chhabra and Soland (1980).

For the test problem with $m=5$ , $n=4$ , and $p=8$ . runs 4a, 4b, and 4c are the same except for the different intermediate steps' option

TABLE 4

ZIPCAP TEST RESULTS

| Problem Size | | | | | Capacitated/ Uncapacitated | Data Information | Run Number | CODE USED | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | RIP30C | ZIPCAP | | | | | | | |
| | | | | | | | | Elapsed Time in Seconds | Options | | | | | | Elapsed Time in Seconds | Number of Nodes |
| m | n | p | Number of Variables | Number of Constraints | | | | | IINPT | ICAPR | ISTEP | IUNCAP | EPS | ET | | |
| 3 | 4 | 5 | [17 | 9] | Capacitated | | | | | | | | | | | |
| | | | | | $s_k$ = 400,400, 1000,400,400 | As given in the other document | 1 | 0.987 | 1 | 1 | 0 | 0 | 0.0 | | 0.017 | 3 |
| | | | | | $s_k$ = 700 $\forall k$ | -,,- | 2 | 0.679 | 1 | 1 | 0 | 0 | 0.0 | | 0.035 | 9 |
| | | | | | $s_k$ = 3000 $\forall k$ | -,,- | 3 | 1.144 | 1 | 1 | 0 | 0 | 0.0 | | 0.018 | 3 |
| 5 | 4 | 8 | [28 | 12] | Capacitated | Given in Appendix B | 4a | 2.421 | 1 | 1 | 0 | 0 | 0.0 | | 0.082 | 9 |
| | | | | | | | 4b | | 1 | 1 | 1 | 0 | 0.0 | | 0.144 | 9 |
| | | | | | | | 4c | | 1 | 1 | 2 | 0 | 0.0 | | 2.033 | 9 |
| | | | | | | | 4d | | 1 | 0 | 0 | 0 | 0.0 | | 0.124 | 19 |
| 10 | 8 | 8 | [38 | 16] | Uncapacitated | As given in the other document | 5 | 485.8 | 1 | 0 | 0 | 1 | 0.0 | | 0.529 | 23 |
| 10 | 30 | 8 | [308 | 38] | Uncapacitated | | 6a | | 1 | 0 | 0 | 1 | 0.002 | 55.0 | 8.229 | 125 |
| | | | | | | | 6b | | 1 | 0 | 0 | 1 | 0.0 | 55.0 | 19.397 | 277 |

(ISTEP) and this results in slight differences in the time taken to solve the problem. Runs 4a and 4d differ in that 4d does not use the capacity rule; the resulting difference in the total number of nodes explored to reach the optimal value points to the effectiveness of the capacity rule in conjunction with the bounding rule.

Run 5 shows the results for an uncapacitated problem with m=10 , n=8 , and p=8 . Option ICAPR is not used since the capacity rule is not useful for an uncapacitated problem.

Another uncapacitated problem with m=10 , n=30 , and p=8 is solved in runs 6a and 6b. In run 6a, the epsilon value (EPS) is specified as 0.002. The solution value found by exploring 125 nodes may be suboptimal but is guaranteed to be within +0.2 percent of the optimal solution value. Run 6b is made with an epsilon value (EPS) of 0.0, and the optimal solution value is found in 277 nodes. A comparison of runs 6a and 6b shows that the number of nodes is less than half for a solution value that may be suboptimal but very close to the optimal solution value as compared to the number of nodes for an optimal solution value.

In general, a small difference between a solution value that may be suboptimal and the optimal solution value, translates into a significant difference in the corresponding number of nodes and the solution time required.

# 6. FURTHER CONSIDERATIONS

It was mentioned in Chapter 2 that it is possible to consider alternative formulations of problem (P), and also to consider choices of Lagrange multipliers other than $v_k = b_k$ with the purpose of obtaining "tighter" bounds which, in turn, would further improve the efficiency of the branch-and-bound procedure. These aspects will be discussed in this Chapter.

## 6.1 Alternative Formulations

Problem (P) can be reformulated by adding additional constraints such that the corresponding Lagrangian relaxation(s), if solved, would provide "tighter" bounds. If such a relaxation does not possess the integrality property, then it provides an equal or better bound compared to that from an LP relaxation, as mentioned in Chapter 2.

Two alternative formulations of problem (P), along with their Lagrangian relaxations, are given below.

### 6.1.1 Alternative Formulation 1

Formulation (AP1) is obtained by adding the constraints $e_{ik}\, x_{ij} \leq y_k$ , for all i,j, and k , to problem (P), i.e.,

$$
(\mathrm{AP1})\left\{
\begin{array}{lllr}
\text{Minimize} & \sum_i \sum_j a_{ij}\, x_{ij} + \sum_k b_k y_k & & (4) \\
\text{subject to} & \sum_i x_{ij} = 1 & \forall j & (2) \\
 & \sum_i \sum_j r_{ijk}\, x_{ij} \leq y_k & \forall k & (5') \\
 & e_{ik}\, x_{ij} \leq y_k & \forall i,j,k & (52) \\
 & x_{ij}\, ,\ y_k = 0 \text{ or } 1 & \forall i,j,k & (6)
\end{array}
\right.
$$

Since $e_{ik} = 1$ or 0, each constraint of (52) is either equivalent to $x_{ij} \leq y_k$ (if $e_{ik} = 1$) or else is redundant (if $e_{ik} = 0$). Problem (API) thus has, at most, mnp additional constraints relative to problem (P). Two Lagrangian relaxations are now considered for problem (AP1).

The first Lagrangian relaxation is obtained with respect to constraints (5') by introducing nonnegative Lagrange multipliers $v_k \geq 0$ to get

$$\text{Minimize} \quad \sum_i \sum_j a_{ij} x_{ij} + \sum_k b_k y_k - \sum_k v_k \left(y_k - \sum_i \sum_j r_{ijk} x_{ij}\right)$$

subject to (2), (52), and (6), or equivalently,

$$(\text{ALR1}_v) \begin{cases} \text{Minimize} & \sum_i \sum_j x_{ij} \left(a_{ij} + \sum_k v_k r_{ijk}\right) - \sum_k y_k \left(v_k - b_k\right) & & (53) \\ \text{subject to} & \sum_i x_{ij} = 1 & \forall j & (2) \\ & e_{ik} x_{ij} \leq y_k & \forall i,j,k & (52) \\ & x_{ij}, y_k = 0 \text{ or } 1 & \forall i,j,k & (6) \end{cases}$$

Another Lagrangian relaxation of problem (AP1) is obtained with respect to constraints (5') and (52) by introducing nonnegative Lagrange multipliers $v_k$ and $\lambda_{ijk}$, respectively, to get

$$\begin{aligned} \text{Minimize} \quad & \sum_i \sum_j a_{ij} x_{ij} + \sum_k b_k y_k \\ & - \sum_k v_k \left(y_k - \sum_i \sum_j r_{ijk} x_{ij}\right) \\ & - \sum_i \sum_j \sum_k \lambda_{ijk} \left(y_k - e_{ik} x_{ij}\right) \end{aligned}$$

subject to (2) and (6), or equivalently,

$$
(\text{ALR1}_{v,\lambda})
\begin{cases}
\text{Minimize} & \sum_i \sum_j x_{ij} \left(a_{ij} + \sum_k v_k \, r_{ijk} + \sum_k e_{ik} \, \lambda_{ijk}\right) \\
& - \sum_k y_k \left(v_k + \sum_i \sum_j \lambda_{ijk} - b_k\right) & (54) \\
\text{Subject to} & \sum_i x_{ij} = 1 \qquad \forall j & (2) \\
& x_{ij} \, , \; y_k = 0 \text{ or } 1 \qquad \forall i,j,k & (6)
\end{cases}
$$

For this problem, the solution is:

$$
\begin{aligned}
y_k &= 0 \quad \text{if} \quad \left(v_k + \sum_i \sum_j \lambda_{ijk} - b_k\right) \leq 0 \, , \\
&= 1 \quad \text{if} \quad \left(v_k + \sum_i \sum_j \lambda_{ijk} - b_k\right) \geq 0 \, ,
\end{aligned}
$$

and $x_{ij} = 1$ if $i$ minimizes $\left(a_{\underline{\ell}j} + \sum_k v_k \, r_{\underline{\ell}jk} + \sum_k e_{\underline{\ell}k} \, \lambda_{\underline{\ell}jk}\right)$

over $\underline{\ell}$ .

We need good choices of Lagrange multipliers $v_k$ with which to solve problem ($\text{ALR1}_v$), and of Lagrange multipliers $v_k$ and $\lambda_{ijk}$ with which to solve problem ($\text{ALR1}_{v,\lambda}$). Problem ($\text{ALR1}_v$) does not possess the integrality property, thus offering the hope of a tight bound, but has more constraints and is difficult to solve compared to problem ($\text{ALR1}_{v,\lambda}$) which, on the other hand, involves more Lagrange multipliers.

### 6.1.2 Alternative Formulation 2

Another formulation of problem (P) is similar to problem (AP1) except for a modification in constraints (5'), i.e.,

$$\text{(AP2)}\begin{cases} \text{Minimize} \quad \sum_i \sum_j a_{ij}\, x_{ij} + \sum_k b_k y_k & (4) \\ \text{Subject to} \quad \sum_i x_{ij} = 1 \qquad \forall j & (2) \\ \sum_i \sum_j r_{ijk}\, x_{ij} \leq 1 \qquad \forall k & (55) \\ e_{ik}\, x_{ij} \leq y_k \qquad \forall i,j,k & (52) \\ x_{ij}\,,\ y_k = 0 \text{ or } 1 \quad \forall i,j,k & (6) \end{cases}$$

A Lagrangian relaxation with respect to constraints (55) and (52) is obtained by introducing nonnegative Lagrange multipliers $v_k$ and $\lambda_{ijk}$ to get

$$\text{Minimize} \quad \sum_i \sum_j a_{ij}\, x_{ij} + \sum_k b_k y_k$$

$$- \sum_k v_k \left(1 - \sum_i \sum_j r_{ijk}\, x_{ij}\right)$$

$$- \sum_i \sum_j \sum_k \lambda_{ijk} \left( y_k - e_{ik}\, x_{ij}\right)$$

Subject to (2) and (6), or equivalently,

$$(\text{ALR2}_{v,\lambda})\begin{cases} \text{Minimize} \quad \sum_i \sum_j x_{ij} \left(a_{ij} + \sum_k v_k\, r_{ijk} + \sum_k e_{ik}\, \lambda_{ijk}\right) & \\ \quad + \sum_k y_k \left(b_k - \sum_i \sum_j \lambda_{ijk}\right) - \sum_k v_k & (56) \\ \text{Subject to} \quad \sum_i x_{ij} = 1 \qquad \forall j & (2) \\ x_{ij}\,,\ y_k = 0 \text{ or } 1 \qquad \forall i,j,k & (6) \end{cases}$$

For this problem, the solution is:

$$y_k = 0 \quad \text{if} \quad \sum_i \sum_j \lambda_{ijk} \leq b_k\,,$$

$$= 1 \quad \text{if} \quad \sum_i \sum_j \lambda_{ijk} \geq b_k\,,$$

and $x_{ij} = 1$ if $i$ minimizes $\left(a_{\ell j} + \sum_k v_k r_{\ell jk} + \sum_k e_{\ell k} \lambda_{\ell jk}\right)$ over $\ell$ .

Here again, we need good choices of the Lagrange multipliers $v_k$ and $\lambda_{ijk}$ with which to solve problem ($ALR2_{v,\lambda}$).

6.1.3 Choice of Lagrange Multipliers

Each of the relaxations ($ALR1_{v,\lambda}$) and ($ALR2_{v,\lambda}$) involves $p$ $v_k$ Lagrange multipliers and $mnp$ $\lambda_{ijk}$ multipliers. If we have good choices of these multipliers, the resulting solutions of the relaxed problems should provide "tighter" bounds (because of the additional constraints) than the bound from relaxation ($LR_v$). Since relaxations ($ALR1_{v,\lambda}$) and ($ALR2_{v,\lambda}$) are similar to a great extent, only the relaxation ($ALR1_{v,\lambda}$) will be considered for further discussion.

By looking at expression (54) of the formulation ($ALR1_{v,\lambda}$), a meaningful choice of the Lagrange multipliers $v_k$ and $\lambda_{ijk}$ appears to follow from setting

$$v_k + \sum_i \sum_j \lambda_{ijk} = b_k \qquad \forall k \tag{57}$$

so that each of the $\lambda_{ijk}$ can be chosen as

$$\left.\begin{aligned} \lambda_{ijk} &= \frac{b_k - v_k}{n\left(\sum_i e_{ik}\right)} \qquad \text{if } e_{ik} = 1 \,, \\ &= 0 \quad \text{otherwise} \end{aligned}\right\} \tag{58}$$

The solution for problem ($ALR1_{v,\lambda}$) is then to select, from each column $j$ , an $x_{ij}$ variable which minimizes $\left(a_{\ell j} + \sum_k v_k r_{\ell jk} + \sum_k e_{\ell k} \lambda_{\ell jk}\right)$ over $\ell$ .

Arbitrary values were considered for the $v_k$ (e.g., $v_k$ equal to $3/4\ b_k$ , $1/2\ b_k$ , $1/4\ b_k$ , and $0$) , the $\lambda_{ijk}$ were then computed from

(58), and the test problem with three designs (m), four activities (n) and five facilities (p) was solved. Three cases with different capacities $s_k$ (as specified in Chapter 5, Table 4) were tried for the solutions at the initial node. The results, however, were not conclusive in terms of providing a meaningful choice of the Lagrange multipliers $v_k$ (and of the $\lambda_{ijk}$).

Since the relaxation ($ALR1_{v,\lambda}$) possesses the integrality property, a choice of the multipliers as the optimal values of the dual variables of the corresponding linear programming problem would provide a solution as good as the LP solution (as stated in Chapter 2). We do not propose to solve linear programs as a part of our branch-and-bound methodology. However, we have made some LP runs, basically to see if the results provide insight leading to the choice of the Lagrange multipliers, and also to see if the resulting LP solutions are "close" to the integer solutions. These results are given below.

The LP formulation ($\overline{AP1}$) corresponding to problem (AP1) is:

$$(\overline{AP1})\begin{cases} \text{Minimize} & \sum_i \sum_j a_{ij}\, x_{ij} + \sum_k b_k y_k & & (4) \\ \text{Subject to} & \sum_i x_{ij} = 1 & \forall j & (2) \\ & \sum \sum r_{ijk}\, x_{ij} \le y_k & \forall k & (5') \\ & e_{ik}\; x_{ij} \le y_k & \forall i,j,k & (52) \\ & y_k \le 1 & \forall k & (15) \\ & x_{ij}\, ,\, y_k \ge 0 & \forall i,j,k & (16) \end{cases}$$

The constraints $x_{ij} \le 1$ are implicit in constraints (2).

Problem ($\overline{AP1}$) was solved for the test problem with $m=3$, $n=4$, and $p=5$ and three different cases for the capacities $s_k$ (as specified in

Chapter 5, Table 4). Each case was solved using the IMSL (International Mathematical and Statistical Library) Code ZX3LP on an IBM 3031 at The George Washington University.

Note that the formulation $(\overline{AP1})$ has up to mnp more constraints than the LP formulation $(\bar{P})$ given in Chapter 2. For our test problem, this translates into solving a problem of 17 variables and 50 constraints corresponding to formulation $(\overline{AP1})$ as against 17 variables and 14 constraints corresponding to formulation $(\bar{P})$.

Table 5 lists the solution values for each of the three cases with different capacities for the small problem with three designs, four activities and five facilities. The solutions to problems $(\bar{P})$ and $(\overline{AP1})$, obtained from ZX3LP, show the optimal solution values, the optimal values of the variables $x_{ij}$ and $y_k$, and the optimal values of the dual variables corresponding to the Lagrangian relaxations $(LR_v)$ and $(ALR1_{v,\lambda})$, i.e., $v_k$ associated with the capacity constraints (3') and $\lambda_{ijk}$ associated with the constraints (52). The table also shows Z(P), and the Lagrangian solution value $Z(LR_v)$ obtained by setting $v_k = b_k$ for all $k$ at the root node, i.e., $Z(LR_b)$.

As expected, the LP solutions for each of the three cases show $Z(\overline{AP1})$ to be considerably higher than $Z(\bar{P})$, and closer to Z(P), thereby providing a tighter bound. As for the Lagrange multipliers $v_k$ and $\lambda_{ijk}$, the following relationships are observed.

$$\sum_i \sum_j \lambda_{ijk} \leq b_k \qquad \forall k \text{, and}$$

$$v_k + \sum_i \sum_j \lambda_{ijk} \geq b_k \qquad \forall k \text{.}$$

Also, for $v_k = 0$, $\sum_i \sum_j \lambda_{ijk} = b_k$, and

for $v_k \geq b_k$, $\sum_i \sum_j \lambda_{ijk} = 0 \quad \forall k$.

TABLE 5

LP AND OTHER SOLUTION VALUES FOR A TEST PROBLEM
(m=3, n=4, AND p=5)

| Test Problem Case | Z(P) | $Z(\bar{P})$ | $Z(LR_b)$ | $Z(\overline{AP1})$ |
|---|---|---|---|---|
| $s_k = 700\ \forall k$ | 37,774.0<br>$x_{21}=1$ $x_{22}=1$ $x_{23}=1$ $x_{24}=1$<br>$y_1=1$ $y_3=1$ $y_5=1$ | 36,688.04<br>$x_{11}=0.2$ $x_{32}=1$ $x_{23}=1$ $x_{24}=1$<br>$x_{21}=0.8$<br>$y_1=0.85$ $y_2=0.05$ $y_3=1$<br>$y_4=0.05$ $y_5=0.85$<br>$v_1=1750$ $v_2=2000$ $v_3=2221$<br>$v_4=1350$ $v_5=1000$ | 36,504.92<br>$x_{11}=1$ $x_{32}=1$ $x_{23}=1$ $x_{24}=1$<br>Solution infeasible in problem (P) | 37,678.14<br>$x_{11}=0.05$ $x_{22}=0.95$ $x_{23}=0.95$ $x_{24}=0.95$<br>$x_{21}=0.95$ $x_{32}=0.05$ $x_{33}=0.05$ $x_{34}=0.05$<br>$y_1=0.95$ $y_2=0.05$ $y_3=0.95$ $y_4=0.05$ $y_5=0.95$<br>$v_1=0$ $v_2=0$ $v_3=1491.2$ $v_4=0$ $v_{[illegible]}=0$<br>$\lambda_{211}=377.2$ $\lambda_{332}=1621.9$ $\lambda_{233}=258.8$ $\lambda_{324}=838.9$ $\lambda_{2[illegible]}=1000.0$<br>$\lambda_{231}=366.8$ $\lambda_{342}=378.1$ $\lambda_{344}=511.1$<br>$\lambda_{241}=1$[illegible]$6.0$ |
| $s_k=400,400$ 1000,400, 400 | 40,174.0<br>$x_{11}=1$ $x_{32}=1$ $x_{33}=1$ $x_{24}=1$<br>$y_1=1$ $y_2=1$ $y_3=1$ $y_4=1$<br>$y_5=1$ | 37,472.62<br>$x_{11}=1$ $x_{32}=1$ $x_{23}=0.8$ $x_{14}=0.2$<br>$x_{33}=0.2$ $x_{24}=0.8$<br>$y_1=1$ $y_2=0.2$ $y_3=1$ $y_4=0.2$<br>$y_5=1$<br>$v_1=1750$ $v_2=2000$ $v_3=5144$<br>$v_4=1350$ $v_5=3879$ | 36,775.5<br>$x_{11}=1$ $x_{32}=1$ $x_{13}=1$ $x_{24}=1$<br>Solution infeasible in problem (P) | 38,360.26<br>$x_{11}=1$ $x_{22}=0.74$ $x_{23}=0.74$ $x_{14}=0.19$<br>$x_{32}=0.26$ $x_{33}=0.26$ $x_{24}=0.55$<br>$x_{34}=0.26$<br>$y_1=1$ $y_2=0.26$ $y_3=1$ $y_4=0.26$ $y_5=1$<br>$v_1=457[illegible].0$ $v_2=0$ $v_3=5063.0$ $v_4=0$ $v_5=$[illegible]$0.0$<br>$\lambda_{322}=553.0$ $\lambda_{324}=657.0$<br>$\lambda_{332}=1447.0$ $\lambda_{344}=693.0$ |
| $s_k=3000\ \forall k$ | 37,429.0<br>$x_{11}=1$ $x_{22}=1$ $x_{23}=1$ $x_{24}=1$<br>$y_1=1$ $y_3=1$ $y_5=1$ | 33,156.2<br>$x_{11}=1$ $x_{32}=1$ $x_{23}=1$ $x_{24}=1$<br>$y_1=0.15$ $y_2=0.01$ $y_3=0.32$<br>$y_4=0.01$ $y_5=0.15$<br>$v_1=1750$ $v_2=2000$ $v_3=1750$<br>$v_4=1350$ $v_5=1000$ | 33,156.2<br>$x_{11}=1$ $x_{32}=1$ $x_{23}=1$ $x_{24}=1$ | 36,754.0<br>$x_{11}=0.5$ $x_{22}=0.5$ $x_{13}=0.5$ $x_{24}=0.5$<br>$x_{21}=0.5$ $x_{32}=0.5$ $x_{23}=0.5$ $x_{34}=0.5$<br>$y_1=0.5$ $y_2=0.5$ $y_3=0.5$ $y_4=0.5$ $y_5=0.5$<br>$v_1=0$ $v_2=0$ $v_3=0$ $v_4=0$<br>$\lambda_{221}=915.0$ $\lambda_{322}=1412.0$ $\lambda_{113}=1020.0$ $\lambda_{324}=1350.0$<br>$\lambda_{231}=28.0$ $\lambda_{342}=588.0$ $\lambda_{213}=675.0$<br>$\lambda_{241}=807.0$ $\lambda_{223}=55.0$<br>$v_5=0$<br>$\lambda_{225}=1000.0$ |

Although the relationship among various $\lambda_{ijk}$ values is not apparent, the above observations are useful in further exploring some good choices of the Lagrange multipliers for the relaxation ($ALR1_{v,\lambda}$), as discussed earlier.

As for the "closeness" of the LP solution to that of the integer solution, most of the solution values $x_{ij}$ and $y_k$ of problem ($\overline{AP1}$) are fractional, and their rounding off to 0 or 1 does not, in general, seem to correspond to the optimal integer solution values $x_{ij}$ and $y_k$ of problem (P).

Table 5 also displays $Z(LR_b)$ at the root node for each of the three cases. For $s_k = 3000$, $Z(LR_b) = Z(\bar{P})$, and the Lagrange multipliers, as reflected by the values of the dual variables of problem ($\bar{P}$), are equal to $b_k$ for all $k$. This is expected from Theorem 3 and our discussion of the integrality property in Chapter 2. Further, the dual variables of ($\bar{P}$) for the first two cases (i.e., $s_k = 700\ \forall k$, and $s_k = 400, \ldots$) have values $v_k \geq b_k$ from Theorem 1.

The $Z(LR_b)$ values in Table 5, however, take no consideration of the capacity rule and/or the facility usage rule of our branch-and-bound procedure. These rules, by fixing certain $x_{ij}$ values to 0 or 1, and by forcing certain facilities into the solution, provide an improved lower bound. As per our branch-and-bound procedure the improved lower bound at the root node is obtained by solving problem ($PR_1$). For example, for $s_k = 700\ \forall k$, the values of $Z(LR_b)$ and $Z(PR_1)$ are shown in Figure 8. The figure also shows the values of $Z(\bar{P})$, $Z(\overline{AP1})$, and $Z(P)$. The branch-and-bound procedure rules provide an improved value of the lower bound $Z(PR_1)$ compared to $Z(\bar{P})$. It appears that some good values of the Lagrange multipliers of the relaxation ($ALR1_{v,\lambda}$), if found, could, in conjunction with these rules, provide significant improvement over $Z(\overline{AP1})$, and without the need to solve an LP problem.

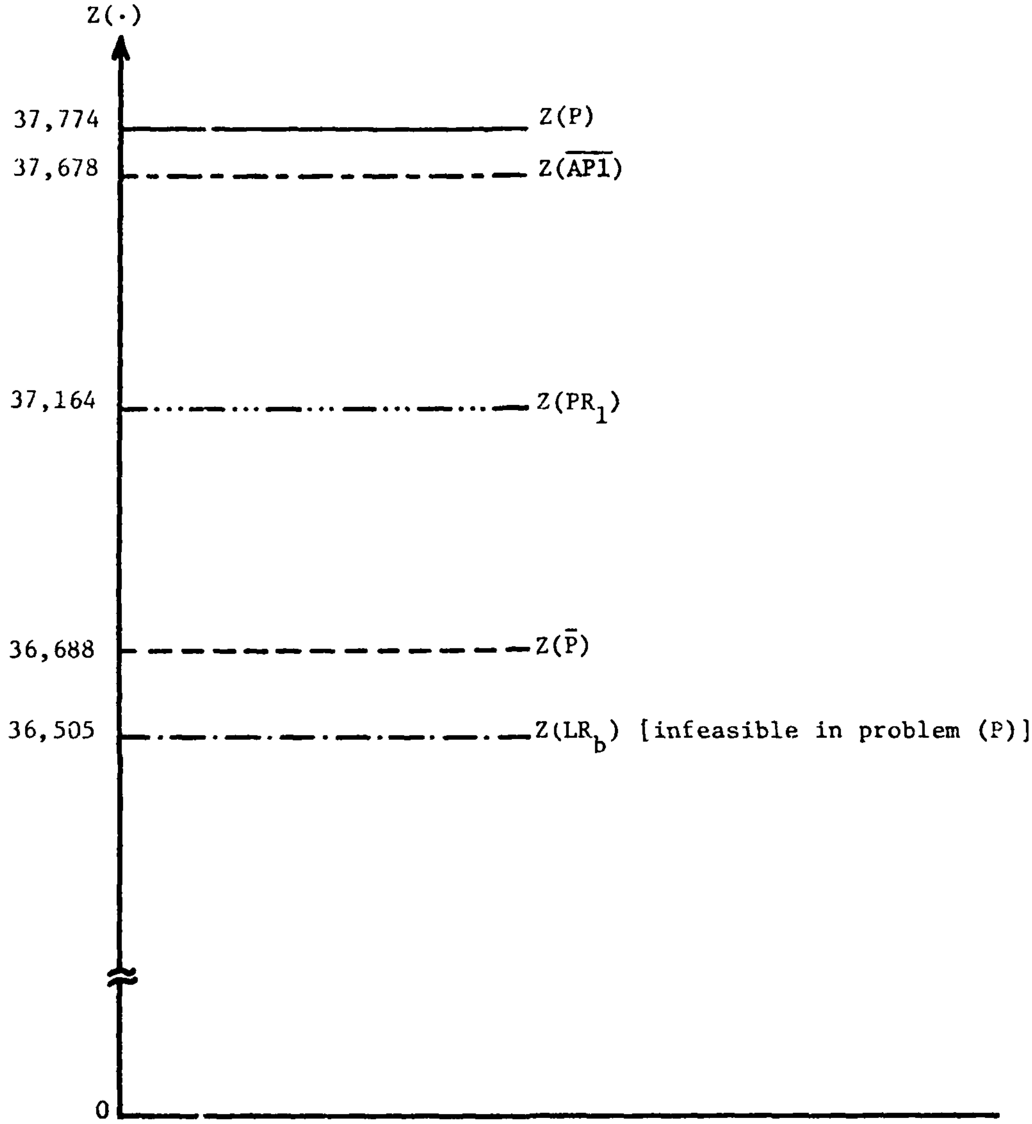



Figure 8

Lagrangian and other solution values for a test problem
(Test problem with m=3, n=4, p=5, and $s_k$ = 700 $\forall k$)

## 6.2 Subgradient Method

It was mentioned in Chapter 2 that setting the Lagrange multipliers $v_k$ equal to $b_k$ for all $k$ provides a good starting point in solving the Lagrangian relaxation $(LR_v)$ of problem (P). From Theorem 3, this choice is optimal (in terms of providing the tightest lower bound) if the resulting solution is feasible in problem (P). In other cases, i.e., where the resulting solution is not feasible in problem (P), this choice is generally not optimal and it is possible to tighten the bounds by considering values of $v_k \geq b_k$ (from Theorem 1). One method that seems useful in providing such a choice is the subgradient method.

The subgradient method is an adaptation of the gradient method in which gradients are replaced by subgradients. Through a heuristic choice of the step-size, this method has been successfully used to provide improved bounds and sometimes optimal solutions [for details see Held, Wolfe, and Crowder (1974), Fisher (1978), and Christofides (1980)]. The fundamental theoretical result is that

$$Z(LR_v^{\,g}) \rightarrow Z(D) \quad \text{if} \quad t^g \rightarrow 0 \quad \text{and} \quad \sum_{q=0}^{g} t^q \rightarrow \infty \quad \text{as} \quad g \rightarrow \infty ,$$

where $t^g$ is the positive step-size $t$ for the gth iteration, and $Z(LR_v^{\,g})$ is the solution value of the relaxed problem $(LR_v)$ using $v_k$ values obtained at the gth iteration.

In the case of problem (P), the step-size $t^{g+1}$ for iteration $g + 1$, given that we have a solution of $(LR_v^{\,g})$, is given by

$$t^{g+1} = \frac{\lambda^{g+1}[Z^* - Z(LR_v^{\,g})]}{\sum_k ||y_k^{\,g} - \sum_i \sum_j r_{ijk}\, x_{ij}^{\,g}||^2} , \tag{59}$$

where $\lambda^{g+1}$ is a scalar and generally between 0 and 2, and $Z^*$ is an upper bound on $Z(LR_v^{\,g})$, frequently obtained by applying a heuristic to solve problem (P).

Given the vector of multipliers $v^g$ , $v^{g+1}$ is generated by

$$v_k^{g+1} = v_k^g - t^{g+1} \left( y_k^g - \sum_i \sum_j r_{ijk} x_{ij}^g \right) , \qquad (60)$$

where we enforce $v_k^{g+1} \geq b_k$ in our case of problem (P) (because of Theorem 1).

Justification for these rules and computational results of applications of the subgradient method are given in Held et al (1974). The scalar $\lambda$ is generally initiated by setting $\lambda^1 = 2$ and halving subsequent $\lambda$'s whenever the resulting solution value has failed to increase in some fixed number of iterations. This rule has performed well empirically [Held et al (1974) and Fisher (1978)].

For the test problem with three designs, four activities, five facilities, and the capacities $s_1 = 400$, $s_2 = 400$, $s_3 = 1000$, $s_4 = 400$, $s_5 = 400$, the Lagrangian solution obtained at the root node by setting $v_k = b_k$ for all $k$ , i.e., the solution to problem ($PR_1$) is infeasible in problem (P), i.e., it violates the capacity constraints. It seems that the subgradient method could be useful in considering $v_k \geq b_k$ with the ultimate purpose of obtaining a tighter lower bound, and, depending on the revised solution(s), possible improvement in the best upper bound. Another possiblity could be to first arbitrarily increase the relevant $v_k$ values by a small percentage of the $b_k$ values and then solve problem ($LR_v$), hopefully to improve the lower bound; and thereafter to use the subgradient method for obtaining subsequent values of $v_k$ , and tightening the bounds.

Both of the areas discussed above, i.e., the consideration of alternative formulations of problem (P), and the application of the subgradient method, and their combination, seem useful for continued research in terms of further improving the branch-and-bound procedure for solving the multiactivity multifacility capacity-constrained 0-1 assignment problems.

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# APPENDIX A

## ZIPCAP LISTING (REVISED)

```
FORTRAN IV G LEVEL  21                 MAIN              DATE = 80315         11/07/24
                 C                   ZIPCAP, A ZERO-ONE INTEGER PROGRAM IS DESIGNED       00000010
                 C                   TO SOLVE MULTIACTIVITY MULTIFACILITY CAPACITY-       00000015
                 C                   CONSTRAINED PROBLEMS HAVING VARIABLE AND FIXED       00000020
                 C                   COSTS. IT ALSO SOLVES UNCAPACITATED PROBLEMS AS A    00000030
                 C                   SPECIAL CASE                                         00000040
 0001                  INTEGER    D(35,35,30), A(35,35), CX(35,35), E(35,30),             00000050
                      1           B(30), BSOLX(35), BSOLY(30), FLB(30),FIX(35),FIXI(35),  00000060
                      2           FUB(30), S(30), SOLX(35), STX(1225)                     00000070
 0002                  REAL                MINC(35), NMINC(35)                            00000080
 0003                  DIMENSION C(35,35), DIFBR(35), KT2(35), MIND(35)                   00000090
 0004                  INTEGER    BR0, BR1, FC, FCUB, P                                   00000110
 0005                  REAL       LOWB, MAXDIF, MINSC                                     00000120
                 C     **************OPTIONS AVAILABLE: IINPT, ICAPR, ISTEP, IUNCAP,EPS   00000130
                 C                   IINPT=1 IF INPUT LISTING DESIRED; 0 OTHERWISE        00000140
                 C                   ICAPR=1 IF CAPACITY RULE TO BE USED; 0 OTHERWISE     00000150
                 C                   ISTEP=0 IF LISTING OF INTERMEDIATE STEPS             00000160
                 C                   NOT DESIRED. ISTEP=1 IF SUMMARY OF BRANCH &          00000170
                 C                   BOUND NODES DESIRED. ISTEP=2 IF DETAILED             00000180
                 C                   LISTING OF INTERMEDIATE STEPS DESIRED.               00000190
                 C                   IUNCAP=1 IF SOLVING AN UNCAPACITATED PROBLEM,        00000200
                 C                   0 OTHERWISE.                                         00000210
                 C                   EPS= A FRACTIONAL VALUE IF SUBOPTIMAL                00000220
                 C                   SOLUTION DESIRED, E.G., EPSILON AS 0.005             00000230
                 C                   IMPLIES SOLUTION TO BE WITHIN +0.5 PERCENT           00000240
                 C                   OF THE OPTIMAL SOLUTION. EPS=0.0 IF OPTIMAL          00000250
                 C                   SOLUTION DESIRED.                                    00000253
                 C                   ET= ELAPSED TIME IN SECONDS, IF SPECIFIED, AT        00000256
                 C                   WHICH THE NODE AND BOUNDS RELATED INFORMATION        00000260
                 C                   IS PRINTED. THIS IS USEFUL IN A SITUATION IF         00000263
                 C                   ISTEP=0 AND THE PROGRAM TERMINATES BEFORE            00000266
                 C                   REACHING THE FINAL SOLUTION.                         00000270
                 C     **************READ INPUT DATA*********************************     00000273
 0006                  READ 10, IINPT, ICAPR, ISTEP, IUNCAP, EPS, ET                      00000280
 0007               10 FORMAT (4I1, F6.5, F10.3)                                          00000290
                 C                   M= NUMBER OF DESIGNS                                 00000300
                 C                   N= NUMBER OF ACTIVITIES                              00000310
                 C                   P= NUMBER OF FACILITIES                              00000320
 0008                  READ 20,M,N,P                                                      00000330
 0009               20 FORMAT (3I5)                                                       00000340
                 C                   A(I,J): VARIABLE COST MATRIX                         00000350
 0010                  READ 30,  ((A(I,J), I=1,M),J=1,N)                                  00000360
 0011               30 FORMAT   (8I10)                                                    00000370
                 C                   B(K): FIXED COST VECTOR                              00000380
 0012                  READ 30, (B(K),K=1,P)                                              00000390
 0013                  IF (IUNCAP.EQ.1) GO TO 40                                          00000400
                 C                   S(K): CAPACITY LIMIT VECTOR; REQUIRED ONLY           00000410
                 C                   IF IUNCAP=0                                          00000420
 0014                  READ 30, (S(K),K=1,P)                                              00000430
                 C                   D(I,J,K): CAPACITY USAGE MATRIX; REQUIRED            00000440
                 C                   ONLY IF IUNCAP=0                                     00000450
 0015                  DO 32 K=1,P                                                        00000460
 0016                  READ 30,((D(I,J,K), I=1,M),J=1,N)                                  00000470
 0017               32 CONTINUE                                                           00000480
 0018                  DO 37 K=1,P                                                        00000490
 0019                  DO 37 I=1,M                                                        00000500
 0020                     IF (D(I,1,K).EQ.0) GO TO 35                                     00000510
 0021                     E(I,K)=1                                                        00000520
 0022                     GO TO 37                                                        00000530
```

AD-A102 583 GEORGE WASHINGTON UNIV WASHINGTON DC PROGRAM IN LOGISTICS F/G 12/1
SOLVING MULTIACTIVITY MULTIFACILITY CAPACITY-CONSTRAINED 0-1 AS--ETC(U)
MAY 81 K L CHHABRA N00014-80-C-0169
UNCLASSIFIED SERIAL-T-441 NL

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FORTRAN IV G LEVEL  21                MAIN              DATE = 80315          11/07/24

0023          35    E(I,K)=0
0024          37 CONTINUE
0025             GO TO 90
            C                   E(I,K): DESIGN-FACILITY MATRIX; REQUIRED ONLY
            C                   IF IUNCAP=1
0026          40 READ 45,((E(I,K),I=1,M),K=1,P)
0027          45 FORMAT  (80I1)
0028             DO 80 K=1,P
0029                S(K)=N
0030             DO 75 I=1,M
0031                IF (E(I,K).EQ.1) GO TO 65
0032             DO 60 J=1,N
0033                D(I,J,K)=0
0034          60 CONTINUE
0035                GO TO 75
0036          65 DO 70 J=1,N
0037                D(I,J,K)=1
0038          70 CONTINUE
0039          75 CONTINUE
0040          80 CONTINUE
            C    **************PRINT INPUT DATA*********************************
0041          90 PRINT 95, IINPT,  ICAPR, ISTEP, IUNCAP, EPS, ET
0042          95 FORMAT ( '1',' OPTIONS SELECTED :  IINPT=',I1,
               1         ' ICAPR=',I1, '  ISTEP=',I1, '  IUNCAP=',I1,
               2         ' EPS=',F8.5, '  ET=', F10.3///)
0043             IF (IINPT.EQ.0) GO TO 168
0044             PRINT  100,M,N,P
0045         100 FORMAT ('0', T55, 'INPUT DATA',/1X, T55, '----- -----',//////1X,
               1T41, 'NUMBER OF DESIGNS (M)=', 4X,I4//1X,T41,
               2'NUMBER OF ACTIVITIES (N)=', 1X,I4//1X, T41,
               3'NUMBER OF FACILITIES (P)=',1X, I4///)
0046             PRINT  105
0047         105 FORMAT ( 4X,    'VARIABLE COST MATRIX  A(I,J)',/4X,
               1'-----------------------------',/)
0048             DO  110 I=1,M
0049         110 PRINT 115, I, (A(I,J),J=1,N)
0050         115 FORMAT ('0', T6, 'I=', I3, 4X,8I13, 4(/, 14X,8I13))
0051             PRINT 120
0052         120 FORMAT('0',//4X,'FIXED COST VECTOR  B(K)',/4X,
               1'----------------------',      /)
0053             PRINT 122, (B(K),K=1,P)
0054         122 FORMAT ('0', T15, 8I13, 3(/, 14X,8I13))
0055             PRINT  125
0056         125 FORMAT('0',//4X,'CAPACITY LIMIT VECTOR  S(K)',/4X,
               1'--------------------------',  /)
0057             PRINT 128, (S(K),K=1,P)
0058         128 FORMAT ('0', T15, 8I13, 3(/, 14X,8I13))
0059             PRINT 130
0060         130 FORMAT('0',//4X, 'CAPACITY USAGE MATRIX  D(I,J,K)',/4X,
               1'-------------------------------',/)
0061             DO  150 K=1,P
0062             PRINT  135,K
0063         135 FORMAT ('0',//5X,'K=',I3/)
0064             DO  145 I=1,M
0065             PRINT  140,I,(D(I,J,K), J=1,N)
0066         140 FORMAT ('0', T6, 'I=', I3, 4X,8I13, 4(/, 14X,8I13))
0067         145 CONTINUE
0068         150 CONTINUE
```

```
FORTRAN IV G LEVEL  21                  MAIN               DATE = 80315         11/07/24

0069              PRINT 155                                                          00001120
0070          155 FORMAT('0',//4X,'DESIGN-FACILITY MATRIX E(I,K)',/4X,               00001130
             1'---------------------------',/)                                       00001140
0071              DO 160  I=1,M                                                      00001150
0072              PRINT 158, I, (E(I,K),K=1,P)                                       00001160
0073          158 FORMAT ('0', T6, 'I=', I3, 4X,8I13, 3(/, 14X,8I13))                00001170
0074          160 CONTINUE                                                           00001180
0075          168 IF (ISTEP.EQ.0) GO TO 190                                          00001190
0076              IF (ISTEP.EQ.1) GO TO 175                                          00001200
0077              PRINT 170                                                          00001210
0078          170 FORMAT ('0',///55X,'DETAILED LISTING OF STEPS',/)                  00001220
0079              GO TO 190                                                          00001230
0080          175 PRINT 180                                                          00001240
0081          180 FORMAT ('0',///55X,'SUMMARY OF STEPS',/)                           00001250
            C     *************INITIALIZE******************************************  00001260
0082          190 BUB=9999999.                                                       00001270
0083              BUBS= BUB/ (1.0+EPS)                                               00001280
0084              NSX=0                                                              00001290
0085              NOD=1                                                              00001310
0086              IBNOD=1                                                            00001315
0087              INET=0                                                             00001320
0088              INSET=0                                                            00001330
0089              DO 205 J=1,N                                                       00001390
0090                 FIX(J)=0                                                        00001400
0091                 KT2(J)=0                                                        00001410
0092              DO 205 I=1,M                                                       00001420
0093                 CX(I,J)=0                                                       00001430
0094          205 CONTINUE                                                           00001433
0095              LQ1=0                                                              00001436
0096              LQ2=0                                                              00001440
0097              LR2=0                                                              00001443
0098              CALL TIMET(ITO)                                                    00001445
0099              IF (ISTEP.EQ.0) GO TO 208                                          00001448
0100              PRINT 220,NOD                                                      00001450
0101          208 IF(NSX.EQ.0) GO TO 283                                             00001453
            C                 CX(I,J) CONTAINS FIXED AND FREE X(I,J) VARIABLES.      00001456
            C                  STX(INS) CONTAINS FIXED X(I,J) VARIABLES.             00001460
            C                  CX(I,J) AND STX(INS) ARE UPDATED BY THE CAPACITY      00001480
            C                  RULE, THE BOUNDING RULE, AND THE RULE FOR             00001490
            C                  BRANCHING AND BACKTRACKING.                           00001500
            C                  IN CX(I,J) A FIXED VARIABLE IS RECORDED AS 1 OR       00001505
            C                  2, AND A FREE VARIABLE AS 0.                          00001510
            C                  A VALUE OF 1 IMPLIES THAT THAT PARTICULAR VARIABLE    00001515
            C                  IS FIXED, AND FIX(J) IS SET EQUAL TO 1 IMPLYING       00001520
            C                  THAT COLUMN J HAS A FIXED VARIABLE OF VALUE 1.        00001525
            C                  A VALUE OF 2 IMPLIES THAT THAT PARTICULAR VARIABLE    00001530
            C                  SHOULD NOT BE CONSIDERED FOR CURRENT COMPUTATIONS.    00001535
            C                  AN X(I,J) RECORDED IN CX(I,J) AS 1 DUE TO THE         00001540
            C                  BRANCHING RULE IS RECORDED IN STX(INS) AS X*100+J.    00001545
            C                  AN X(I,J) RECORDED IN CX(I,J) AS 1 DUE TO THE         00001550
            C                  CAPACITY RULE OR THE BOUNDING RULE IS RECORDED IN     00001555
            C                  STX(INS) AS (X*100+J)+1000000.                        00001560
            C                  AN X(I,J) RECORDED IN CX(I,J) AS 2 IS RECORDED IN     00001565
            C                  STX(INS) AS -(X*100+J)-1000000.                       00001570
0102          210 IF (ISTEP.EQ.0) GO TO 225                                          00001580
0103          215 PRINT 220,NOD                                                      00001590
0104          220 FORMAT ('0',//6X,'NODE NUMBER', I7/)                               00001600
            C     **************UPDATE CX(I,J) FOR BRO*****************************  00001610
```

```
              C                       BRO IS THE RIGHT BRANCHING VARIABLE             00001615
  0105          225 LX=BRO                                                            00001620
  0106              IX=LX/100                                                         00001630
  0107              JX=LX-IX*100                                                      00001640
  0108              CX(IX,JX)=2                                                       00001650
  0109              KT2(JX)=KT2(JX)+1                                                 00001660
  0110              FIX(JX)=0                                                         00001720
  0111              LQ1=LQ1-1                                                         00001725
  0112              IF (KT2(JX).LT.(M-1)) GO TO 270                                   00001730
  0113              DO 255 I=1,M                                                      00001740
  0114                 IF (CX(I,JX).EQ.2) GO TO 255                                   00001750
  0115                 CX(I,JX)=1                                                     00001760
  0116                 NSX=NSX+1                                                      00001763
  0117                 STX(NSX)=    (I*100+JX)+1000000                                00001766
  0118                 FIX(JX)=1                                                      00001770
  0119                 LQ1=LQ1+1                                                      00001780
  0120                 FIXI(JX)=I                                                     00001790
  0121                 GO TO 270                                                      00001800
  0122          255 CONTINUE                                                          00001810
  0123          270 LQ2=0                                                             00001820
  0124              LR2=0                                                             00001825
  0125              GO TO 283                                                         00001830
  0126          272 IF (ISTEP.EQ.0) GO TO 276                                         00001840
  0127              PRINT 220,NOD                                                     00001850
              C     **************UPDATE CX(I,J) FOR BR1*******.*******************   00001853
              C                       BR1 IS THE LEFT BRANCHING VARIABLE              00001856
  0128          276 LQ2=0                                                             00001860
  0129              LR2=0                                                             00001866
  0130              LX=BR1                                                            00001870
  0131              IX=LX/100                                                         00001875
  0132              JX=LX-IX*100                                                      00001880
  0133              CX(IX,JX)=1                                                       00001885
  0134              FIX(JX)=1                                                         00001890
  0135              LQ1=LQ1+1                                                         00001892
  0136              DO 279 I=1,M                                                      00001895
  0137                 IF (IX.EQ.I) GO TO 281                                         00001897
  0138          279 CONTINUE                                                          00001900
  0139          281 FIXI(JX)=IX                                                       00001902
  0140          283 IF (ISTEP.NE.2) GO TO 303                                         00001905
  0141          285 DO 295 I=1,M                                                      00001910
  0142                 PRINT 290, I,(CX(I,J),J=1,N)                                   00001920
  0143          290    FORMAT (/5X,'CX(I,J)',4X,'I=',I3,2X, 20I4/23X, 20I4)           00001930
  0144          295 CONTINUE                                                          00001940
  0145              PRINT 297,(FIX(J),J=1,N)                                          00001950
  0146          297 FORMAT (/5X,'FIX(J)',12X, 20I4/23X, 20I4)                         00001960
              C     **************APPLY CAPACITY RULE******************************   00001970
              C                       AND UPDATE CX(I,J) AND STX(INS).                00001980
  0147          303 DO 307 K=1,P                                                      00002000
  0148                 FLB(K)=0                                                       00002015
  0149          307 CONTINUE                                                          00002025
  0150          310 DO 2000 K=1,P                                                     00002030
              C                       FIND THE SUM OF MINIMUM D(I,J,K) OVER EACH J FOR A  00002040
              C                       GIVEN K, I.E., MINSD= SUM OF MIND(J)            00002050
  0151              MINSD=0                                                           00002060
  0152              DO 900 J=1,N                                                      00002070
  0153                 IF(FIX(J).EQ.0) GO TO 350                                      00002080
              C                       IF  FIX(J)=1, SET MIND(J)=D(I,J,K) FOR CX(I,J)=1    00002090
              C                       AND MOVE TO NEXT COLUMN J                       00002100
```

```
                INDI=FIXI(J)
                MIND(J)=D(INDI,J,K)
                GO TO 800
          350      LK=0
                   I=1
                   MIND(J)=D(I,J,K)
        C                     SKIP D(I,J,K) WHEN CX(I,J)=2 & MOVE TO NEXT ROW I
          400      IF(CX(I,J).EQ.2) GO TO 600
          500      IF(D(I,J,K).LT.MIND(J)) MIND(J)=D(I,J,K)
                   GO TO 700
          600      LK=LK+1
                   IF(I.GT.LK) GO TO 700
                   I=I+1
                   MIND(J)=D(I,J,K)
                   GO TO 750
          700      I=I+1
          750      IF(I.LE.M) GO TO 400
          800      MINSD=MINSD+MIND(J)
          900 CONTINUE
          910 IF (ISTEP.NE.2) GO TO 960
                   PRINT 950, K, MINSD,(MIND(J),J=1,N)
          950      FORMAT ('0','K,MINSD,(MIND(J),J=1,N)',10I10,4(/,44X,8I10))
          960      IF (MINSD.EQ.0) GO TO 975
          965      IF( FLB(K).EQ.1) GO TO 975
          970      FLB(K)=1
          975      IF (IUNCAP.EQ.1) GO TO 2000
          978      IF (ICAPR.EQ.0) GO TO 2000
        C                     FIND BALANCE AVAILABLE CAPACITY IBALD FOR A GIVEN K
        C                     IF IBALD IS NEGATIVE, THEN BACKTRACK.
          980      IBALD=S(K)-MINSD
         1000 IF (IBALD.LT.0) GO TO 6200
                DO 1500 J=1,N
        C                     SKIP COLUMN J IF FIX(J)=1
                   IF (FIX(J).EQ.1) GO TO 1500
                DO 1300 I=1,M
        C                     SKIP ROW I IF CX(I,J)=2
         1100      IF(CX(I,J).EQ.2) GO TO 1300
        C                     COMPUTE DIFFERENCE BETWEEN D(I,J,K) AND MIND(J).
        C                     IF IT IS MORE THAN AVAILABE BALANCE, SET CX(I,J)=2
         1200      IDIFD=D(I,J,K)-MIND(J)
                   IF ((IDIFD-IBALD).LE.0)  GO TO 1300
                   CX(I,J)=2
                   NSX=NSX+1
                   STX(NSX)=-(I*100+J)-1000000
        C                     LQ2 COUNTS THE NUMBER OF CX(I,J) VALUES SET EQUAL
        C                     TO 2 IN A CYCLE
                   LQ2=LQ2+1
        C                     KT2(J) KEEPS AN ACCOUNT OF CX(I,J) VALUES SET EQUAL
        C                     TO 2 FOR COLUMN J
                   KT2(J)=KT2(J)+1
        C                     FOR COLUMN J, IF ALL BUT ONE CX(I,J) VALUES ARE
        C                     EQUAL TO 2, SET THAT CX(I,J)=1 &  SET FIX(J)=1
                   IF(KT2(J).LT.(M-1)) GO TO 1300
                DO 1250 LR=1,M
                   IF(CX(LR,J).EQ.2) GO TO 1250
                   CX(LR,J)=1
                   NSX=NSX+1
                   STX(NSX)=   (LR*100+J)+1000000
```

```
FORTRAN IV G LEVEL  21                  MAIN               DA E = 80315          11/07/24
0199                   FIX(J)=1                                                         00002650
              C                  LQ1 KEEPS AN ACCOUNT OF COLUMNS FOR WHICH FIX(J)=1    00002655
0200                   LQ1=LQ1+1                                                        00002660
              C                  FIXI(J) SPECIFIES INDEX I FOR WHICH FIX(J)=1          00002662
0201                   FIXI(J)=LR                                                       00002665
0202                   GO TO 1500                                                       00002670
0203            1250 CONTINUE                                                           00002680
0204            1300 CONTINUE                                                           00002690
0205            1500 CONTINUE                                                           00002700
0206            1800 IF (ISTEP.NE.2) GO TO 2000                                         00002710
0207                 PRINT 1900, K, LQ2, LQ1                                            00002720
0208            1900 FORMAT ('0','K=',I3,'  LQ2=',I3, '  LQ1=', I3)                     00002730
0209                 DO 1930 I=1,M                                                      00002740
0210                   PRINT 290, I,(CX(I,J),J=1,N)                                     00002750
0211            1930 CONTINUE                                                           00002770
0212                 PRINT 297,  (FIX(J),J=1,N)                                         00002780
0213            2000 CONTINUE                                                           00002800
              C                  A CYCLE EXAMINES ALL THE FACILITIES.                  00002803
              C                  IF IN A CYCLE, THE CAPACITY RULE RESULTS IN SETTING   00002810
              C                  ADDITIONAL CX(I,J) VALUES EQUAL TO 2, THEN REPEAT     00002820
              C                  THE CYCLE. BUT IF FIX(J)=1 FOR ALL J, THEN DO NOT     00002830
              C                  REPEAT THE CYCLE.                                     00002835
0214                 IF (LQ1.EQ.N) GO TO 2400                                           00002840
0215                 IF (LQ2.EQ.LR2) GO TO 2400                                         00002845
0216            2200 LR2=LQ2                                                            00002860
0217                 GO TO 310                                                          00002870
              C      **************SOLVE (LAGRANGIAN) RELAXED PROBLEM**************     00002880
              C                 UPDATE VECTOR OF FACILITIES FLB(K) FOR COMPUTING       00002890
              C                  C(I,J) MATRIX & LOWER BOUND. IT HAS VALUE 1 IF A      00002900
              C                  FACILITY IS USED, OTHERWISE IT HAS 0 VALUE.           00002910
0218            2400 DO 3000 J=1,N                                                      00002950
0219                   IF (FIX(J).EQ.0) GO TO 3000                                      00002960
0220                   INDI=FIXI(J)                                                     00002970
0221                 DO 2550 K=1,P                                                      00002990
0222                   IF (E(INDI,K).EQ.0) GO TO 2550                                   00003000
0223                   IF (FLB(K).EQ.1) GO TO 2550                                      00003010
0224                   FLB(K)=1                                                         00003020
0225            2550 CONTINUE                                                           00003030
0226            3000 CONTINUE                                                           00003060
0227                 IF (ISTEP.NE.2) GO TO 3150                                         00003070
0228                 PRINT 3100, (FLB(K),K=1,P)                                         00003080
0229            3100 FORMAT('0','(FLB(K),K=1,P)  ',  20I4/16X,20I4)                     00003090
              C                  COMPUTE COST MATRIX C(I,J) FOR THE RELAXED PROBLEM    00003100
0230            3150 DO 3400 J=1,N                                                      00003110
0231                 DO 3300 I=1,M                                                      00003120
0232                   BSUM=0.0                                                         00003130
0233                 DO 3200 K=1,P                                                      00003140
0234                   IF (FLB(K).EQ.1) GO TO 3200                                      00003150
0235                   IF (E(I,K).EQ.0) GO TO 3200                                      00003160
0236                   BSUM=BSUM+(B(K) * (FLOAT(D(I,J,K))/ FLOAT(S(K))))                00003170
0237            3200 CONTINUE                                                           00003180
0238            3250   C(I,J)=A(I,J)+BSUM                                               00003190
0239            3300 CONTINUE                                                           00003200
0240            3400 CONTINUE                                                           00003210
0241                 IF (ISTEP.NE.2) GO TO 3445                                         00003220
0242                 DO 3430 I=1,M                                                      00003230
0243                   PRINT 3420, I, (C(I,J),J=1,N)                                    00003250
0244            3420   FORMAT (/5X, 'C(I,J)',5X, 'I=', I3,2X, 5F15.4,                   00003260
```

```
                   1          6(/23X, 5F15.4))                                     00003265
 0245          3430 CONTINUE                                                       00003270
              C                    FIND SUM OF MINIMUM C(I,J) VALUES OVER EACH J,  00003290
              C                    I.E., MINSC=SUM OF MINC(J).                     00003300
              C                    IF FIX(J)=1, THEN MINC(J)=C(I,J) WHERE CX(I,J)=1 00003310
 0246          3445 MINSC=0.0                                                      00003320
 0247               DO 3900 J=1,N                                                  00003340
 0248                  IF (FIX(J).EQ.0) GO TO 3500                                 00003350
 0249                  INDI=FIXI(J)                                                00003360
 0250                  MINC(J)=C(INDI,J)                                           00003370
 0251                  SOLX(J)=INDI                                                00003380
 0252                  GO TO 3850                                                  00003410
 0253          3500    LK=0                                                        00003430
 0254                  I=1                                                         00003440
              C                    SKIP C(I,J) ELEMENT IF CX(I,J)=2 & MOVE TO NEXT I 00003470
 0255          3550    IF (CX(I,J).EQ.2) GO TO 3700                                00003480
 0256                  IF ((I-LK).EQ.1) GO TO 3600                                 00003485
 0257                  IF (C(I,J).GE.MINC(J)) GO TO 3750                           00003490
 0258          3600    MINC(J)=C(I,J)                                              00003500
 0259                  IMIN=I                                                      00003510
 0260                  GO TO 3750                                                  00003520
 0261          3700    LK=LK+1                                                     00003530
 0262          3750    I=I+1                                                       00003590
 0263          3800    IF (I.LE.M) GO TO 3550                                      00003600
 0264                  SOLX(J)=IMIN                                                00003610
 0265          3850    MINSC=MINSC+MINC(J)                                         00003620
 0266          3900 CONTINUE                                                       00003630
 0267               IF (ISTEP.NE.2) GO TO 3940                                     00003640
 0268               DO 3920 J=1,N                                                  00003650
 0269                  PRINT 3910, J,MINC(J),SOLX(J)                               00003660
 0270          3910 FORMAT ('0','J,MINC(J),SOLX(J)', I5,F15.4,I6)                  00003670
 0271          3920 CONTINUE                                                       00003680
              C                    COMPUTE FIXED COST FC FOR LOWER BOUND           00003710
 0272          3940 FC=0                                                           00003720
 0273               DO 4000 K=1,P                                                  00003730
 0274                  IF (FLB(K).EQ.0) GO TO 4000                                 00003740
 0275          3950    FC=FC+B(K)                                                  00003750
 0276          4000 CONTINUE                                                       00003760
              C     **************COMPUTE LOWER BOUND LOWB************************ 00003770
 0277          4050 LOWB=MINSC+FC                                                  00003780
 0278               IF (ISTEP.EQ.0) GO TO 4150                                     00003790
 0279               PRINT 4120, MINSC, FC, LOWB                                    00003800
 0280          4120 FORMAT ('0',' MINSC, FC, LOWB ', F15.4, I15, F15.4)            00003810
              C                    COMPARE LOWER BOUND WITH BEST UPPER BOUND STAR  00003820
              C                    BUBS WHICH EQUALS BUB/(1+EPS). IF LOWB IS       00003830
              C                    GREATER THAN OR EQUAL TO BUBS, THEN BACKTRACK   00003840
 0281          4150 IF (LOWB.GE.BUBS)GO TO 6200                                    00003850
              C                    CHECK IF CURRENT SOLUTION SATISFIES CAPACITY    00003880
              C                    CONSTRAINTS                                     00003890
 0282          4200 IF (IUNCAP.EQ.1) GO TO 4420                                    00003900
 0283          4210 DO 4400 K=1,P                                                  00003910
 0284                  NSUMD=0                                                     00003920
 0285               DO 4300 J=1,N                                                  00003930
 0286                  IX=SOLX(J)                                                  00003950
 0287                  NSUMD=NSUMD+D(IX,J, K)                                      00003960
 0288          4300 CONTINUE                                                       00003970
 0289               IF (ISTEP.NE.2) GO TO 4320                                     00003980
 0290                  PRINT 4310, K,NSUMD                                         00003990
```

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 0291          4310     FORMAT ('0', 'K,NSUMD',2I10)                                  00004000
 0292          4320     IF(NSUMD.LE.S(K)) GO TO 4400                                  00004010
 0293                   GO TO 5100                                                    00004020
 0294          4400 CONTINUE                                                          00004030
             C      **************COMPUTE UPPER BOUND UPB IF CAPACITY CONSTRAINTS     00004040
             C                    ARE SATISFIED.                                      00004050
             C                    UPB=SUM OF A(I,J)+FIXED COST FCUB BASED ON          00004060
             C                    SOLUTION VECTOR SOLX(J)                             00004070
             C                    VECTOR OF FACILITIES FOR UPPER BOUND FUB(K) HAS     00004080
             C                    VALUES 1 OR 0 BASED ON FACITILY USED OR OTHERWISE   00004090
 0295          4420 DO 4450 K=1,P                                                     00004100
 0296                   FUB(K)=0                                                      00004110
 0297          4450 CONTINUE                                                          00004120
 0298               NSUMA=0                                                           00004130
 0299               FCUB=0                                                            00004140
 0300          4500 DO 4650 J=1,N                                                     00OC 4150
 0301                   IX=SOLX(J)                                                    00004170
 0302                   NSUMA=NSUMA+A(IX,J)                                           00004180
 0303          4550 DO 4600 K=1,P                                                     00004190
 0304                   IF(E(IX,K).EQ.0) GO TO 4600                                   00004200
 0305                   IF(FUB(K).EQ.1) GO TO 4600                                    00004210
 0306                   FUB(K)=1                                                      00004220
 0307                   FCUB=FCUB+B(K)                                                00004230
 0308          4600 CONTINUE                                                          00004240
 0309          4650 CONTINUE                                                          00004250
 0310               IF (ISTEP.NE.2) GO TO 4700                                        00004260
 0311               PRINT 4660, (FUB(K),K=1,P)                                        00004270
 0312          4660 FORMAT('0','(FUB(K),K=1,P)   ',  20I4/16X,20I4)                   00004280
 0313          4700 UPB=NSUMA+FCUB                                                    00004290
 0314          4708 IF (ISTEP.EQ.0) GO TO 4750                                        00004300
 0315               PRINT 4710, NSUMA, FCUB, UPB, BUB, BUBS                           00004310
 0316          4710 FORMAT('0','NSUMA, FCUB, UPB, BUB, BUBS ',2I10, 2F15.4)           00004320
             C                    COMPARE UPPER BOUND WITH BEST UPPER BOUND           00004330
             C                    IF UPB IS LESS THAN BUB, SET IT AS BUB AND          00004340
             C                    NOTE THE SOLUTION                                   00004350
 0317          4750 IF (UPB.GE.BUB) GO TO 5100                                        00004360
 0318          4770 BUB=UPB                                                           00004370
 0319               BUBS= BUB/ (1.0+EPS)                                              00004380
 0320               IBNOD=NOD                                                         00004385
 0321               PRINT 4780, IBNOD, BUB, BUBS                                      00004386
 0322          4780 FORMAT ('0', 'IBNOD, BUB, BUBS', I10, 2F15.4)                     00004388
 0323               DO 4800 J=1,N                                                     00004390
 0324          4800 BSOLX(J)=SOLX(J)                                                  00004400
 0325               DO 4850 K=1,P                                                     00004410
 0326          4850 BSOLY(K)=FUB(K)                                                   00004420
             C      **************COMPARE LOWB WITH BUBS. IF LOWB IS GREATER          00004430
             C                    THAN OR EQUAL TO BUBS, THEN BACKTRACK               00004440
 0327          4900 IF (LOWB.GE.BUBS)GO TO 6200                                       00004450
             C      **************IF FIX(J) VALUES ARE 1 FOR EACH J, THEN BACKTRACK   00004480
 0328          5100 IF (LQ1.EQ.N) GO TO 6200                                          00004500
             C      **************APPLY THE BOUNDING RULE*************************    00004510
             C                    IF THE DIFFERENCE BETWEEN C(I,J) AND MINC(J) IS     00004515
             C                    GREATER THAN THE DIFFERENCE BETWEEN BUBS AND        00004520
             C                    LOWB, THEN CX(I,J)=2                                00004525
             C      **************APPLY BRANCHING RULE AND FIND BR1, THE NEXT         00004530
             C                    VARIABLE FOR LEFT BRANCHING.                        00004540
             C                    FIND NMINC(J), THE NEXT HIGHER VALUE THAN MINC(J)   00004550
             C                    AND DIFBR(J), THE DIFFERENCE BETWEEN THEM.          00004555
```

```
            DBOUND=BUBS-LOWB                                        00004568
      5200  DO 5250 J=1,N                                           00004570
               NMINC(J)=0.0                                         00004580
               DIFBR(J)=0.0                                         00004590
      5250  CONTINUE                                                00004600
            DO 5600 J=1,N                                           00004610
C                          SKIP TO NEXT J IF FIX(J)=1               00004620
               IF (FIX(J).EQ.1) GO TO 5600                          00004630
               LK=0                                                 00004640
               I=1                                                  00004650
C                          SKIP C(I,J) IF CX(I,J)=2 & MOVE TO NEXT I 00004670
      5300     IF (CX(I,J).EQ.2) GO TO 5350                         00004680
               IF (I.EQ.SOLX(J)) GO TO 5350                         00004690
               IF ((C(I,J)-MINC(J)).GT.DBOUND) GO TO 5330           00004700
               IF ((I-LK).EQ.1) GO TO 5320                          00004710
               IF (C(I,J).GE.NMINC(J)) GO TO 5400                   00004720
      5320     NMINC(J)=C(I,J)                                      00004730
               GO TO 5400                                           00004735
      5330     CX(I,J)=2                                            00004740
               NSX=NSX+1                                            00004742
               STX(NSX)=-(I*100+J)-1000000                          00004745
               KT2(J)=KT2(J)+1                                      00004747
               IF(KT2(J).LT.(M-1)) GO TO 5350                       00004750
               INDI=SOLX(J)                                         00004752
               CX(INDI,J)=1                                         00004755
               NSX=NSX+1                                            00004758
               STX(NSX)= (INDI*100+J)+1000000                       00004760
               FIX(J)=1                                             00004762
               LQ1=LQ1+1                                            00004764
               FIXI(J)=INDI                                         00004766
               GO TO 5600                                           00004768
      5350     LK=LK+1                                              00004770
      5400     I=I+1                                                00004775
               IF(I.LE.M) GO TO 5300                                00004780
      5500     DIFBR(J)=NMINC(J)-MINC(J)                            00004785
      5600  CONTINUE                                                00004790
            IF (ISTEP.NE.2) GO TO 5650                              00004795
            DO 5620 I=1,M                                           00004820
               PRINT 290, I,(CX(I,J),J=1,N)                         00004830
      5620  CONTINUE                                                00004850
            PRINT 297,  (FIX(J),J=1,N)                              00004860
C                          IF FIX(J)=1 FOR ALL J, THEN BACKTRACK.   00004880
      5650  IF (LQ1.EQ.N) GO TO 6200                                00004890
C                          FIND MAXDIF, THE MAXIMUM DIFFERENCE DIFBR(J) 00004900
            LF=0                                                    00004905
            DO 5800 J=1,N                                           00004910
               IF (FIX(J).EQ.1) GO TO 5690                          00004915
               IF ((J-LF).EQ.1) GO TO 5660                          00004920
               IF (DIFBR(J).LT.MAXDIF) GO TO 5800                   00004925
      5660     MAXDIF=DIFBR(J)                                      00004930
               LJ=J                                                 00004935
               GO TO 5800                                           00004940
      5690  LF=LF+1                                                 00004943
      5800  CONTINUE                                                00004946
            IF (ISTEP.NE.2) GO TO 5840                              00004950
            DO 5820 J=1,N                                           00004953
               IF (FIX(J).EQ.1) GO TO 5820                          00004956
               PRINT 5810, J, NMINC(J), MINC(J), DIFBR(J)           00004960
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 0383           5810 FORMAT ('0','J,NMINC(J),MINC(J),DIFBR(J)', I5,3F15.4)            00004963
 0384           5820 CONTINUE                                                         00004966
               C     *************BRANCHING VARIABLE BR1 CORRESPONDS TO MAXDIF*****   00004970
 0385           5840 DO 5900 J=1,N                                                    00004980
 0386                   IF (J.NE.LJ) GO TO 5900                                       00004990
 0387           5850    BR1=SOLX(J)*100+J                                             00005000
 0388                IF (ISTEP.EQ.0) GO TO 6020                                       00005010
 0389                   PRINT 5880,  BR1                                              00005020
 0390           5880    FORMAT('0',' BR1',I10)                                        00005030
 0391                   GO TO 6020                                                    00005040
 0392           5900 CONTINUE                                                         00005050
               C     **************UPDATE STX(INS) AND NSX************************    00005060
               C                     NSX REPRESENTS THE NUMBER OF VARIABLES IN STX(INS)  00005070
 0393           6020 NSX=NSX+1                                                        00005090
 0394           6040 STX(NSX)=BR1                                                     00005100
 0395                IF (ISTEP.NE.2) GO TO 6100                                       00005150
 0396                PRINT 6088, (STX(INS), INS=1,NSX)                                00005160
 0397           6088 FORMAT('0',' STX(INS)', 10I10, 122(/, 10X,10I10))                00005170
               C     **************MOVE TO THE NEXT (LEFT BRANCH) NODE AND APPLY      00005220
               C                     CAPACITY RULE                                    00005230
 0398           6100 NOD=NOD+1                                                        00005240
 0399           6110 IF (ET.EQ.0.0) GO TO 6150                                        00005242
 0400                IF(INSET.EQ.1) GO TO 6147                                        00005244
 0401                IF (INET.EQ.1) GO TO 6150                                        00005246
 0402                CALL TIMET(INT)                                                  00005248
 0403                ELTN=(INT-ITO)*26.04E-6                                          00005250
 0404                IF (ELTN.LT.ET) GO TO 6150                                       00005253
 0405           6120 PRINT 6125, NOD, ELTN, BUB, BUBS, IBNOD                          00005256
 0406           6125 FORMAT ('0', 'WAS AT NODE',I6, ' AT ELAPSED TIME =', F10.4,      00005260
                    1        ' SECONDS.',/1X, '  BUB=',F15.4, '  BUBS=',F15.4,        00005263
                    2        '  AT NODE=',I7)                                         00005266
 0407                IBUB=BUB                                                         00005267
 0408                IF (IBUB.EQ.9999999) GO TO 6146                                  00005268
 0409           6130 PRINT 6135, (BSOLX(J),J=1,N)                                     00005270
 0410           6135 FORMAT('0', 'SOLUTION CORRESPONDING TO BUB IS',  //1X,           00005273
                    1        '(BSOLX(J), J=1,N)',10I8,3(/18X,10I8))                   00005276
 0411           6140 PRINT 6145, (BSOLY(K), K=1,P)                                    00005280
 0412           6145 FORMAT(/1X,'(BSOLY(K), K=1,P)',10I8,2(/18X, 10I8))               00005290
 0413           6146 INET=1                                                           00005292
 0414                INIS=ISTEP                                                       00005294
 0415                INSET=1                                                          00005296
 0416                ISTEP=2                                                          00005298
 0417                GO TO 6150                                                       00005300
 0418           6147 ISTEP=INIS                                                       00005302
 0419                INSET=0                                                          00005304
 0420           6150 GO TO 272                                                        00005306
               C     **************END IF AT THE ROOT NODE****************************  00005308
 0421           6200 IF (NSX.EQ.0) GO TO 8100                                         00005310
 0422           6220 IF ( IABS(STX(NSX)).GT.1000000) GO TO 6500                       00005320
 0423           6250 BRO=STX(NSX)                                                     00005330
 0424           6270 STX(NSX)=-BRO-1000000                                            00005340
 0425                IF (ISTEP.EQ.0) GO TO 6308                                       00005390
 0426                PRINT 6305,  BRO                                                 00005400
 0427           6305 FORMAT('0', 'BRO  ',I10)                                         00005410
 0428           6308 IF (ISTEP.NE.2) GO TO 6330                                       00005420
 0429                PRINT 6088, (STX(INS), INS=1,NSX)                                00005430
               C     **************MOVE TO THE NEXT (RIGHT BRANCH) NODE AND APPLY     00005490
               C                     CAPACITY RULE                                    00005500
```

```
0430      6330 NOD=NOD+1                                                    00005510
0431      6410 IF (ET.EQ.0.0) GO TO 6450                                    00005512
0432           IF (INSET.EQ.1) GO TO 6445                                   00005516
0433           IF (INET.EQ.1) GO TO 6450                                    00005518
0434           CALL TIMET(INT)                                              00005520
0435           ELTN=(INT-ITO)*26.04E-6                                      00005523
0436           IF (ELTN.LT.ET) GO TO 6450                                   00005526
0437      6420 PRINT 6125, NOD, ELTN, BUB, BUBS, IBNOD                      00005528
0438           IBUB=BUB                                                     00005530
0439           IF (IBUB.EQ.9999999) GO TO 6442                              00005532
0440      6430 PRINT 6135, (BSOLX(J),J=1,N)                                 00005533
0441      6440 PRINT 6145, (BSOLY(K), K=1,P)                                00005536
0442      6442 INET=1                                                       00005538
0443           INIS=ISTEP                                                   00005540
0444           INSET=1                                                      00005542
0445           ISTEP=2                                                      00005544
0446           GO TO 6450                                                   00005546
0447      6445 ISTEP=INIS                                                   00005548
0448           INSET=0                                                      00005550
0449      6450 GO TO 210                                                    00005552
0450      6500 IF ( STX(NSX).GT.1000000) GO TO 6520                         00005555
0451           LX=-STX(NSX)-1000000                                         00005560
0452           IX=LX/100                                                    00005570
0453           JX=LX-IX*100                                                 00005580
0454           CX(IX,JX)=0                                                  00005590
0455           KT2(JX)=KT2(JX)-1                                            00005595
0456           GO TO 6550                                                   00005600
0457      6520 LX= STX(NSX)-1000000                                         00005610
0458           IX=LX/100                                                    00005620
0459           JX=LX-IX*100                                                 00005630
0460           CX(IX,JX)=0                                                  00005640
0461           FIX(JX)=0                                                    00005650
0462           LQ1=LQ1-1                                                    00005660
0463      6550 NSX=NSX-1                                                    00005690
0464           GO TO 6200                                                   00005700
         C     **************PRINT THE OUTPUT*********************************** 00005730
0465      8100 IBUB=BUB                                                     00005740
0466           CALL TIMET(IT1)                                              00005750
0467           ELT1=(IT1-ITO)*26.04E-6                                      00005760
0468           PRINT 8105, ELT1                                             00005770
0469      8105 FORMAT ('0',///1X, 'ELAPSED TIME IN SECONDS=', F15.8)        00005780
0470           PRINT 8120, NOD                                              00005790
0471      8120 FORMAT ( '0','TOTAL NUMBER OF NODES EXPLORED =',I3)          00005800
0472           IF (IBUB.EQ.9999999) GO TO 8350                              00005810
0473      8130 PRINT 8150                                                   00005820
0474      8150 FORMAT ('0', 'NOTE: 1.  FOLLOWING X(I,J) VARIABLES SHOW DESIGN', 00005830
              1        ' I TO WHICH ACTIVITY J IS ASSIGNED FOR J=1 TO N.',  00005840
              2        /7X, '2.  IF EPSILON EPS WAS ASSIGNED A POSITIVE',   00005850
              3        ' (NON-ZERO) VALUE, THE SOLUTION MAY BE SUBOPTIMAL.',/) 00005860
0475      8180 PRINT 6200, (BSOLX(J),J=1,N)                                 00005870
0476      8200 FORMAT('0',T55,  'OPTIMAL SOLUTION',/1X,T55,                 00005880
              1        '----------------',//1X, 'X(I,J) WITH VALUE 1:',10I8, 00005890
              2        3(/,21X,10I8))                                       00005900
0477      8220 PRINT 8250, (BSOLY(K), K=1,P)                                00005910
0478      8250 FORMAT ('0',      'Y(K) VALUES:', 8X, 10I8, 2(/,21X,10I8))   00005920
0479      8280 PRINT 8300,IBUB                                              00005930
0480      8300 FORMAT ('0',  'OPTIMAL VALUE OF OBJECTIVE FUNCTION:', I15///) 00005940
0481           GO TO 8500                                                   00005950
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 0482          8350 PRINT 8400                                                              00005960
 0483          8400 FORMAT ('0', ' PROBLEM DOES NOT HAVE A FEASIBLE SOLUTION',              00005970
                   1        /1X,    ' BECAUSE THE CAPACITY CONSTRAINTS CANNOT',             00005980
                   2        /1X, ' BE SATISFIED.',/)                                        00005990
 0484          8500 PRINT 8550                                                              00006000
 0485          8550 FORMAT ('0','*****NORMAL END OF JOB*****',/)                            00006010
 0486          8600 STOP                                                                    00006020
 0487               END                                                                     00006030
```

# APPENDIX B

## DETAILED PRINTOUT FOR A TEST PROBLEM
## (TEST PROBLEM WITH m=5, n=4, AND p=8)

OPTIONS SELECTED : IINPT=1 ICAPR=1 ISTEP=2 IUNCAP=0 EPS= 0.0 ET= 0.0

INPUT DATA

NUMBER OF DESIGNS (M)= 5
NUMBER OF ACTIVITIES (N)= 4
NUMBER OF FACILITIES (P)= 8

VARIABLE COST MATRIX A(I,J)

| | | | | |
|---|---|---|---|---|
| I= 1 | 194951 | 218871 | 155094 | 104056 |
| I= 2 | 235277 | 272087 | 143264 | 138641 |
| I= 3 | 196138 | 220718 | 160399 | 107481 |
| I= 4 | 198751 | 224042 | 167046 | 112445 |
| I= 5 | 190873 | 229169 | 128361 | 112498 |

FIXED COST VECTOR B(K)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 14000 | 14000 | 14000 | 14000 | 14000 | 29000 | 19000 | 31000 |

CAPACITY LIMIT VECTOR S(K)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 350 | 350 | 350 | 200 | 200 | 700 | 500 | 800 |

CAPACITY USAGE MATRIX Q(I,J,K)

K= 1

| | | | | |
|---|---|---|---|---|
| I= 1 | 80 | 90 | 100 | 180 |
| I= 2 | 0 | 0 | 0 | 0 |
| I= 3 | 80 | 90 | 100 | 180 |
| I= 4 | 40 | 30 | 50 | 120 |
| I= 5 | 0 | 0 | 0 | 0 |
| K= 2 | | | | |
| I= 1 | 80 | 90 | 100 | 180 |
| I= 2 | 0 | 0 | 0 | 0 |
| I= 3 | 80 | 90 | 100 | 180 |
| I= 4 | 40 | 30 | 50 | 120 |
| I= 5 | 0 | 0 | 0 | 0 |
| K= 3 | | | | |
| I= 1 | 80 | 90 | 100 | 180 |
| I= 2 | 0 | 0 | 0 | 0 |
| I= 3 | 80 | 90 | 100 | 180 |
| I= 4 | 40 | 30 | 50 | 120 |
| I= 5 | 0 | 0 | 0 | 0 |
| K= 4 | | | | |
| I= 1 | 80 | 90 | 100 | 180 |
| I= 2 | 0 | 0 | 0 | 0 |
| I= 3 | 80 | 90 | 100 | 180 |
| I= 4 | 40 | 30 | 50 | 120 |
| I= 5 | 80 | 90 | 100 | 180 |
| K= 5 | | | | |
| I= 1 | 80 | 90 | 100 | 180 |
| I= 2 | 0 | 0 | 0 | 0 |
| I= 3 | 80 | 90 | 100 | 160 |

| | | | | |
|---|---|---|---|---|
| I= 4 | 40 | 30 | 50 | 120 |
| I= 5 | 80 | 90 | 100 | 180 |

K= 6

| | | | | |
|---|---|---|---|---|
| I= 1 | 240 | 180 | 300 | 450 |
| I= 2 | 0 | 0 | 0 | 0 |
| I= 3 | 120 | 90 | 150 | 360 |
| I= 4 | 0 | 0 | 0 | 0 |
| I= 5 | 40 | 30 | 50 | 120 |

K= 7

| | | | | |
|---|---|---|---|---|
| I= 1 | 160 | 180 | 200 | 360 |
| I= 2 | 0 | 0 | 0 | 0 |
| I= 3 | 80 | 60 | 100 | 240 |
| I= 4 | 0 | 0 | 0 | 0 |
| I= 5 | 160 | 180 | 200 | 360 |

K= 8

| | | | | |
|---|---|---|---|---|
| I= 1 | 200 | 150 | 250 | 600 |
| I= 2 | 40 | 30 | 50 | 180 |
| I= 3 | 0 | 0 | 0 | 0 |
| I= 4 | 0 | 0 | 0 | 0 |
| I= 5 | 120 | 90 | 150 | 300 |

DESIGN-FACILITY MATRIX E(I,K)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| I= 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| I= 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| I= 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| I= 4 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |

```
I= 5          0      0      0      1      1      1      1      1

                     DETAILED LISTING OF STEPS

NODE NUMBER      1

   CX(I,J)    I= 1    0  0  0  0
   CX(I,J)    I= 2    0  0  0  0
   CX(I,J)    I= 3    0  0  0  0                                   Step 1
   CX(I,J)    I= 4    0  0  0  0
   CX(I,J)    I= 5    0  0  0  0
   FIX(J)             0  0  0  0
K,MINSD,(MIND(J),J=1,N)     1     0     0     0     0     0          Step 2 - - k=1
K=  1  LQ2=  0  LQ1=  0
   CX(I,J)    I= 1    0  0  0  0
   CX(I,J)    I= 2    0  0  0  0
   CX(I,J)    I= 3    0  0  0  0
   CX(I,J)    I= 4    0  0  0  0
   CX(I,J)    I= 5    0  0  0  0
   FIX(J)             0  0  0  0
K,MINSD,(MIND(J),J=1,N)     2     0     0     0     0     0          Step 2 - - k=2
K=  2  LQ2=  0  LQ1=  0
   CX(I,J)    I= 1    0  0  0  0
   CX(I,J)    I= 2    0  0  0  0
   CX(I,J)    I= 3    0  0  0  0
   CX(I,J)    I= 4    0  0  0  0
   CX(I,J)    I= 5    0  0  0  0
   FIX(J)             0  0  0  0
K,MINSD,(MIND(J),J=1,N)     3     0     0     0     0     0          Step 2 - - k=3
K=  3  LQ2=  0  LQ1=  0
   CX(I,J)    I= 1    0  0  0  0
   CX(I,J)    I= 2    0  0  0  0
```

```
    CX(I,J)    I= 3    0  0  0  0
    CX(I,J)    I= 4    0  0  0  0
    CX(I,J)    I= 5    0  0  0  0
    FIX(J)             0  0  0  0
K,MINSD,(MIND(J),J=1,N)      4     0     0     0     0     0          Step 2 - - k=4
K=  4  LQ2=  0  LQ1=  0
    CX(I,J)    I= 1    0  0  0  0
    CX(I,J)    I= 2    0  0  0  0
    CX(I,J)    I= 3    0  0  0  0
    CX(I,J)    I= 4    0  0  0  0
    CX(I,J)    I= 5    0  0  0  0
    FIX(J)             0  0  0  0
K,MINSD,(MIND(J),J=1,N)      5     0     0     0     0     0          Step 2 - - k=5
K=  5  LQ2=  0  LQ1=  0
    CX(I,J)    I= 1    0  0  0  0
    CX(I,J)    I= 2    0  0  0  0
    CX(I,J)    I= 3    0  0  0  0
    CX(I,J)    I= 4    0  0  0  0
    CX(I,J)    I= 5    0  0  0  0
    FIX(J)             0  0  0  0
K,MINSD,(MIND(J),J=1,N)      6     0     0     0     0     0          Step 2 - - k=6
K=  6  LQ2=  0  LQ1=  0
    CX(I,J)    I= 1    0  0  0  0
    CX(I,J)    I= 2    0  0  0  0
    CX(I,J)    I= 3    0  0  0  0
    CX(I,J)    I= 4    0  0  0  0
    CX(I,J)    I= 5    0  0  0  0
    FIX(J)             0  0  0  0
K,MINSD,(MIND(J),J=1,N)      7     0     0     0     0     0          Step 2 - - k=7
K=  7  LQ2=  0  LQ1=  0
    CX(I,J)    I= 1    0  0  0  0
    CX(I,J)    I= 2    0  0  0  0
    CX(I,J)    I= 3    0  0  0  0
```

```
    CX(I,J)    I=  4     0    0    0    0
    CX(I,J)    I=  5     0    0    0    0
    FIX(J)               0    0    0    0
K,MINSD,(MIND(J),J=1,N)        8        0        0        0        0        0          Step 2 - - k=8
K=  8  LO2=  0  LO1=  0
    CX(I,J)    I=  1     0    0    0    0
    CX(I,J)    I=  2     0    0    0    0
    CX(I,J)    I=  3     0    0    0    0
    CX(I,J)    I=  4     0    0    0    0
    CX(I,J)    I=  5     0    0    0    0
    FIX(J)               0    0    0    0
(FLB(K),K=1,P)     0    0    0    0    0    0    0    0                               Step 3
    C(I,J)     I=  1      239180.9375    262123.4375    210381.4375    205785.9375
    C(I,J)     I=  2      236826.9375    273249.4375    145201.5000    145615.9375
    C(I,J)     I=  3      224777.9375    249997.9375    196198.9375    177800.9375
    C(I,J)     I=  4      209150.9375    231841.9375    180045.9375    143644.9375
    C(I,J)     I=  5      214402.9375    253296.4375    157773.4375    167802.9375
J,MINC(J),SOLX(J)    1    209150.9375     4
J,MINC(J),SOLX(J)    2    231841.9375     4
J,MINC(J),SOLX(J)    3    145201.5000     2
J,MINC(J),SOLX(J)    4    143644.9375     4
 P'NSC, FC, LOWB     729839.3125          0    729839.3125                          Step 4
K,NSUMD         1       190                                                         Step 6
K,NSUMD         2       190
K,NSUMD         3       190
K,NSUMD         4       190
K,NSUMD         5       190
K,NSUMD         6         0
K,NSUMD         7         0
K,NSUMD         8        50
(FUB(K),K=1,P)     1    1    1    1    1    0    0    1                               Step 7
NSUMA, FCUB, UPB, BUB, RUBS     678502    101000    779502.0000    9999999.0000    9999999.0000
IBNOD, RUB, BUBS        1    779502.0000    779502.0000
```

```
    CX(I,J)    I=  1     0   0   2   2                                          Step 11
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   0
    CX(I,J)    I=  4     0   0   0   0
    CX(I,J)    I=  5     0   0   0   0
    FIX(J)               0   0   0   0
J,NMINC(J),MINC(J),DIFBR(J)    1    214402.9375    209150.9375      5252.0000
J,NMINC(J),MINC(J),DIFBR(J)    2    249997.9375    231841.9375     18156.0000
J,NMINC(J),MINC(J),DIFBR(J)    3    157773.4375    145201.5000     12571.9375
J,NMINC(J),MINC(J),DIFBR(J)    4    145615.9375    143644.9375      1971.0000
 BR1       402
 STX(INS)  -1000103  -1000303  -1000104       402

    NODE NUMBER        2

    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   0
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   0
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        1       30        0        30        0        0      Step 2 - - k=1
K=  1  LQ2=  0  LQ1=  1
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   0
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   0
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        2       30        0        30        0        0      Step 2 - - k=2
K=  2  LQ2=  0  LQ1=  1
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
```

```
    CX(I,J)     I=  3     0   0   2   0
    CX(I,J)     I=  4     0   1   0   0
    CX(I,J)     I=  5     0   0   0   0
    FIX(J)                0   1   0   0
K,MINSD,(MIND(J),J=1,N)        3      30      0      30      0      0          Step 2 - - k=3
K=  3  LQ2=  0  LQ1=  1
    CX(I,J)     I=  1     0   0   2   2
    CX(I,J)     I=  2     0   0   0   0
    CX(I,J)     I=  3     0   0   2   0
    CX(I,J)     I=  4     0   1   0   0
    CX(I,J)     I=  5     0   0   0   0
    FIX(J)                0   1   0   0
K,MINSD,(MIND(J),J=1,N)        4      30      0      30      0      0          Step 2 - - k=4
K=  4  LQ2=  2  LQ1=  1
    CX(I,J)     I=  1     0   0   2   2
    CX(I,J)     I=  2     0   0   0   0
    CX(I,J)     I=  3     0   0   2   2
    CX(I,J)     I=  4     0   1   0   0
    CX(I,J)     I=  5     0   0   0   2
    FIX(J)                0   1   0   0
K,MINSD,(MIND(J),J=1,N)        5      30      0      30      0      0          Step 2 - - k=5
K=  5  LQ2=  2  LQ1=  1
    CX(I,J)     I=  1     0   0   2   2
    CX(I,J)     I=  2     0   0   0   0
    CX(I,J)     I=  3     0   0   2   2
    CX(I,J)     I=  4     0   1   0   0
    CX(I,J)     I=  5     0   0   0   2
    FIX(J)                0   1   0   0
K,MINSD,(MIND(J),J=1,N)        6       0      0       0      0      0          Step 2 - - k=6
K=  6  LQ2=  2  LQ1=  1
    CX(I,J)     I=  1     0   0   2   2
    CX(I,J)     I=  2     0   0   0   0
    CX(I,J)     I=  3     0   0   2   2
```

```
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   2
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        7        0        0        0        0        0        Step 2 - - k=7
K=  7  LQ2=  2  LQ1=  1
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   2
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   2
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        8        0        0        0        0        0        Step 2 - - k=8
K=  8  LQ2=  2  LQ1=  1
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   2
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   2
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        1       30        0       30        0        0        Step 2 - - k=1
K=  1  LQ2=  2  LQ1=  1                                                                (Second Cycle)
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   2
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   2
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        2       30        0       30        0        0        Step 2 - - k=2
K=  2  LQ2=  2  LQ1=  1                                                                (Second Cycle)
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   2
    CX(I,J)    I=  4     0   1   0   0
```

```
    CX(I,J)    I=  5     0   0   0   2
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        3       30        0        30        0        0
K=  3  LQ2=  2  LQ1=  1
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   2
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   2
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        4       30        0        30        0        0
K=  4  LQ2=  2  LQ1=  1
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   2
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   2
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        5       30        0        30        0        0
K=  5  LQ2=  2  LQ1=  1
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   2
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   2
    FIX(J)               0   1   0   0
K,MINSD,(MIND(J),J=1,N)        6        0        0         0        0        0
K=  6  LQ2=  2  LQ1=  1
    CX(I,J)    I=  1     0   0   2   2
    CX(I,J)    I=  2     0   0   0   0
    CX(I,J)    I=  3     0   0   2   2
    CX(I,J)    I=  4     0   1   0   0
    CX(I,J)    I=  5     0   0   0   2
```

Step 2 - - k=3
(Second Cycle)

Step 2 - - k=4
(Second Cycle)

Step 2 - - k=5
(Second Cycle)

Step 2 - - k=6
(Second Cycle)

```
   FIX(J)              0   1   0   0
K,MINSC,(MIND(J),J=1,N)        7         0         0         0         0         0        Step 2 - - k=7
K=  7  LQ2=  2  LQ1=  1                                                                   (Second Cycle)
   CX(I,J)    I=  1    0   0   2   2
   CX(I,J)    I=  2    0   0   0   0
   CX(I,J)    I=  3    0   0   2   2
   CX(I,J)    I=  4    0   1   0   0
   CX(I,J)    I=  5    0   0   0   2
   FIX(J)              0   1   0   0
K,MINSD,(MIND(J),J=1,N)        8         0         0         0         0         0        Step 2 - - k=8
K=  8  LQ2=  2  LQ1=  1                                                                   (Second Cycle)
   CX(I,J)    I=  1    0   0   2   2
   CX(I,J)    I=  2    0   0   0   0
   CX(I,J)    I=  3    0   0   2   2
   CX(I,J)    I=  4    0   1   0   0
   CX(I,J)    I=  5    0   0   0   2
   FIX(J)              0   1   0   0
(FLB(K),K=1,P)     1   1   1   1   1   0   0   0                                          Step 3
   C(I,J)     I=  1      218380.9375    238723.4375    184381.4375    158985.9375
   C(I,J)     I=  2      236826.9375    273249.4375    145201.5000    145615.9375
   C(I,J)     I=  3      203977.9375    226597.9375    170198.9375    131000.9375
   C(I,J)     I=  4      198751.0000    224042.0000    167046.0000    112445.0000
   C(I,J)     I=  5      203202.9375    240696.4375    143773.4375    142602.9375
J,MINC(J),SOLX(J)    1    198751.0000     4
J,MINC(J),SOLX(J)    2    224042.0000     4
J,MINC(J),SOLX(J)    3    143773.4375     5
J,MINC(J),SOLX(J)    4    112445.0000     4
 MINSC, FC, LOWB       679011.4375          70000     749011.4375                         Step 4
K,NSUMD          1        190                                                             Step 6
K,NSUMD          2        190
K,NSUMD          3        190
K,NSUMD          4        290
   CX(I,J)    I=  1    0   0   2   2                                                      Step 11
```

Step 11 (continued)

```
   CX(I,J)    I= 2     2   0   0   2
   CX(I,J)    I= 3     0   0   2   2
   CX(I,J)    I= 4     0   1   0   1
   CX(I,J)    I= 5     0   0   0   2
   FIX(J)              0   1   0   1
J,NMINC(J),MINC(J),DIFBR(J)    1    203202.9375    198751.0000     4451.9375
J,NMINC(J),MINC(J),DIFBR(J)    3    145201.5000    143773.4375     1428.0625
BR1      401
STX(INS)  -1000103  -1000303  -1000104      402  -1000304  -1000504  -1000201  -1000204   1000404      401


   NODE NUMBER      3

   CX(I,J)    I= 1     0   0   2   2
   CX(I,J)    I= 2     2   0   0   2
   CX(I,J)    I= 3     0   0   2   2
   CX(I,J)    I= 4     1   1   0   1
   CX(I,J)    I= 5     0   0   0   2
   FIX(J)              1   1   0   1
K,MINSD,(MIND(J),J=1,N)        1       190       40       30        0       120
K=  1  LQ2=  0  LQ1=  3
   CX(I,J)    I= 1     0   0   2   2
   CX(I,J)    I= 2     2   0   0   2
   CX(I,J)    I= 3     0   0   2   2
   CX(I,J)    I= 4     1   1   0   1
   CX(I,J)    I= 5     0   0   0   2
   FIX(J)              1   1   0   1
K,MINSD,(MIND(J),J=1,N)        2       190       40       30        0       120
K=  2  LQ2=  0  LQ1=  3
   CX(I,J)    I= 1     0   0   2   2
   CX(I,J)    I= 2     2   0   0   2
   CX(I,J)    I= 3     0   0   2   2
   CX(I,J)    I= 4     1   1   0   1
   CX(I,J)    I= 5     0   0   0   2
```

Step 2 - - k=1

Step 2 - - k=2

```
    FIX(J)               1   1   0   1
K,MINSD,(MIND(J),J=1,N)        3       190        40        30         0       120        Step 2 - - k=3
K=  3  LC2=  0  LQ1=  3
    CX(I,J)     I=  1    0   0   2   2
    CX(I,J)     I=  2    2   0   0   2
    CX(I,J)     I=  3    0   0   2   2
    CX(I,J)     I=  4    1   1   0   1
    CX(I,J)     I=  5    0   0   0   2
    FIX(J)               1   1   0   1
K,MINSD,(MIND(J),J=1,N)        4       190        40        30         0       120        Step 2 - - k=4
K=  4  LQ2=  2  LQ1=  4
    CX(I,J)     I=  1    0   0   2   2
    CX(I,J)     I=  2    2   0   1   2
    CX(I,J)     I=  3    0   0   2   2
    CX(I,J)     I=  4    1   1   2   1
    CX(I,J)     I=  5    0   0   2   2
    FIX(J)               1   1   1   1
K,MINSD,(MIND(J),J=1,N)        5       190        40        30         0       120        Step 2 - - k=5
K=  5  LQ2=  2  LQ1=  4
    CX(I,J)     I=  1    0   0   2   2
    CX(I,J)     I=  2    2   0   1   2
    CX(I,J)     I=  3    0   0   2   2
    CX(I,J)     I=  4    1   1   2   1
    CX(I,J)     I=  5    0   0   2   2
    FIX(J)               1   1   1   1
K,MINSD,(MIND(J),J=1,N)        6         0         0         0         0         0        Step 2 - - k=6
K=  6  LQ2=  2  LQ1=  4
    CX(I,J)     I=  1    0   0   2   2
    CX(I,J)     I=  2    2   0   1   2
    CX(I,J)     I=  3    0   0   2   2
    CX(I,J)     I=  4    1   1   2   1
    CX(I,J)     I=  5    0   0   2   2
    FIX(J)               1   1   1   1
```

```
K,MINSO,(MINC(J),J=1,N)       7      0      0      0      0      0          Step 2 - - k=7
K=  7  LQ2=  2  LQ1=  4
    CX(I,J)     I=  1     0   0   2   2
    CX(I,J)     I=  2     2   0   1   2
    CX(I,J)     I=  3     0   0   2   2
    CX(I,J)     I=  4     1   1   2   1
    CX(I,J)     I=  5     0   0   2   2
    FIX(J)                1   1   1   1
K,MINSO,(MIND(J),J=1,N)       8     50      0      0     50      0          Step 2 - - k=8
K=  8  LQ2=  2  LQ1=  4
    CX(I,J)     I=  1     0   0   2   2
    CX(I,J)     I=  2     2   0   1   2
    CX(I,J)     I=  3     0   0   2   2
    CX(I,J)     I=  4     1   1   2   1
    CX(I,J)     I=  5     0   0   2   2
    FIX(J)                1   1   1   1
(FLB(K),K=1,P)       1   1   1   1   1   0   0   1                          Step 3
    C(I,J)      I=  1      210630.9375     232910.9375     174693.9375     135735.9375
    C(I,J)      I=  2      235277.0000     272087.0000     143264.0000     138641.0000
    C(I,J)      I=  3      203977.9375     226597.9375     170198.9375     131000.9375
    C(I,J)      I=  4      198751.0000     22[illegible].0000     167046.0000     112445.0000
    C(I,J)      I=  5      198552.9375     237208.9375     137960.9375     130977.9375
J,MINC(J),SOLX(J)     1    19[illegible]       4
J,MINC(J),SOLX(J)     2    224042.0000       4
J,MINC(J),SOLX(J)     3    143264.0000       2
J,MINC(J),SOLX(J)     4    112445.0000       4
 MINSC, FC, LOWB      678502.0000          101000     779502.0000             Step 4
BRO          401                                                              Step 12
 STX(INS)  -1000103  -1000303  -1000104      402  -1000304  -1000504  -1000201  -1000204   1000404  -1000401


     NODE NUMBER        4
     ====================

     CX(I,J)     I=  1     0   0   2   2
```

```
    CX(I,J)    I=  2    2   0   0   2
    CX(I,J)    I=  3    0   0   2   2
    CX(I,J)    I=  4    2   1   0   1
    CX(I,J)    I=  5    0   0   0   2
    FIX(J)              0   1   0   1
K,MINSD,(MIND(J),J=1,N)        1     150       0      30       0     120        Step 2 - - k=1
K=  1  LQ2=  0  LQ1=  2
    CX(I,J)    I=  1    0   0   2   2
    CX(I,J)    I=  2    2   0   0   2
    CX(I,J)    I=  3    0   0   2   2
    CX(I,J)    I=  4    2   1   0   1
    CX(I,J)    I=  5    0   0   0   2
    FIX(J)              0   1   0   1
K,MINSD,(MIND(J),J=1,N)        2     150       0      30       0     120        Step 2 - - k=2
K=  2  LQ2=  0  LQ1=  2
    CX(I,J)    I=  1    0   0   2   2
    CX(I,J)    I=  2    2   0   0   2
    CX(I,J)    I=  3    0   0   2   2
    CX(I,J)    I=  4    2   1   0   1
    CX(I,J)    I=  5    0   0   0   2
    FIX(J)              0   1   0   1
K,MINSD,(MIND(J),J=1,N)        3     150       0      30       0     120        Step 2 - - k=3
K=  3  LQ2=  0  LQ1=  2
    CX(I,J)    I=  1    0   0   2   2
    CX(I,J)    I=  2    2   0   0   2
    CX(I,J)    I=  3    0   0   2   2
    CX(I,J)    I=  4    2   1   0   1
    CX(I,J)    I=  5    0   0   0   2
    FIX(J)              0   1   0   1
K,MINSD,(MIND(J),J=1,N)        4     230      80      30       0     120        Step 2 - - k=4
BRO        402                                                                  Step 12
 STX(INS)  -1000103  -1000303  -1000104  -1000402
```

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```
    C(I,J)       I=  1      204550.9375     229670.9375     167093.9375     125655.9375
    C(I,J)       I=  2      235277.0000     272087.0000     143264.0000     138641.0000
    C(I,J)       I=  3      205737.9375     231517.9375     172398.9375     129080.9375
    C(I,J)       I=  4      203550.9375     227641.9375     173045.9375     126844.9375
    C(I,J)       I=  5      190873.0000     229169.0000     128361.0000     112498.0000
J,MINC(J),SOLX(J)     1     190873.0000       5
J,MINC(J),SOLX(J)     2     229169.0000       5
J,MINC(J),SOLX(J)     3     128361.0000       5
J,MINC(J),SOLX(J)     4     138641.0000       2
 MINSC, FC, LOWB      687044.0000           106000     793044.0000


ELAPSED TIME IN SECONDS=      2.03278542

TOTAL NUMBER OF NODES EXPLORED =  9

NOTE: 1.  FOLLOWING X(I,J) VARIABLES SHOW DESIGN I TO WHICH ACTIVITY J IS ASSIGNED FOR J=1 TO N.
      2.  IF EPSILON EPS WAS ASSIGNED A POSITIVE (NON-ZERO) VALUE, THE SOLUTION MAY BE SUBOPTIMAL.

                                 OPTIMAL SOLUTION

X(I,J) WITH VALUE 1:         4       4       2       4

Y(K) VALUES:                 1       1       1       1       1       0       0       1

OPTIMAL VALUE OF OBJECTIVE FUNCTION:         779502


******NORMAL END OF JOB******
```

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